MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 93 No. 1

MEETING OF 1932 NOVEMBER 11

Dr. H. Knox-Shaw, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

William Thomson Hay, 45 The Chase, Norbury, London, S.W. 16 ; and J. M. Mohr, Physics Dept., University of Bratislava, Czechoslovakia.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

George Barron, B.Sc., The Grammar School, Morley, Leeds (proposed by G. R. Goldsbrough) ;

Frank M. Bateson, 368 Main Road, Karori, Wellington, New Zealand (proposed by R. A. McIntosh) ;

Ernest William Bliss, Demonstrator in Meteorology, Imperial College of Science and Technology, South Kensington, S.W. 7 (proposed by E. B. H. Wade) ;

Keith Edward Bullen, M.A., B.Sc., St. John's College, Cambridge (proposed by H. Jeffreys) ;

John Edmund Calthrop, M.A., M.Sc., Lecturer and Demonstrator in Physics, East London College, Mile End Road, London, E. 1 (proposed by G. S. Le Beau) ;

Don Tudor Ernest Dassanayake, B.Sc., Colombo Observatory, Ceylon (proposed by A. J. Bamford) ;

J. A. Edgar, B.A., Sidney Sussex College, Cambridge (proposed by A. S. Eddington) ;

Major Edward Aubrey Glennie, D.S.O., R.E., Geodetic Branch, Survey of India, Dehra Dun, India (proposed by G. P. Lenox-Conyngham) ;

Garret A. Hobart, 129 Market Street, Paterson, N.J., U.S.A. (proposed by R. L. Waterfield) ;

George Lyman, B.Sc., 21 Holt Road, Blackheath, Birmingham (proposed by M. C. Johnson) ;

Arthur Gordon Nock, B.Sc., Beechcroft, Bloxcidge Street, Langley Green, Birmingham (proposed by J. G. Howarth) ;

Clifford E. Smith, State Teachers College, Fresno, California, U.S.A. (proposed by R. G. Aitken) ;

Bertha Swirles, Ph.D., Department of Mathematics, Imperial College of Science and Technology, South Kensington, London, S.W. 7 (proposed by S. Chapman) ; and

Geoffrey Gilbert Wiles, M.Sc., Balliol College, Oxford (proposed by E. A. Milne).

Two hundred and sixteen presents were announced as having been received since the last meeting, including :—

Bordeaux Observatory, Nouvelles observations des étoiles contenues dans le Second Catalogue de l'Observatoire de Bordeaux ;

Royal Observatory, Cape of Good Hope, Annals, vol. xiii, pt. 2. Discussion of Observations of Occultations of Stars by the Moon, 1672–1908, being a revision of Newcomb's " Researches on the Motion of the Moon, Part II," by H. Spencer Jones ;

D. Dietz, The Story of Science ;

Royal Observatory, Greenwich, Observations of Colour Temperatures of Stars made at the Royal Observatory, Greenwich, in the years 1926–1932 ;

Sir T. L. Heath, Greek Astronomy ;

Radcliffe Observatory, Oxford, The Observations of the Reverend Thomas Hornsby, D.D., Savilian Professor of Astronomy and Radcliffe Observer, made . . . in the years 1774 to 1798. Reduced by H. Knox-Shaw, J. Jackson, and W. H. Robinson.

G. A. Wainwright, A Pair of Constellations (with Map by Miss E. Williamson).

Celestial Photographs.

The following have been added to the list of Celestial Photographs reproduced by the Society for sale to Fellows :—

R.A.S.
Ref. No.

430. Meteor Trail and Pleiades. 1932 Jan. 11. Lockyer.
431. Nebula M 8, N.G.C. 6523. 1919 June 27. Mount Wilson.
432. Nebula M 16, N.G.C. 6611. 1919 Aug. 25–26. Mount Wilson.
433. Cluster in Sagittarius, M 22. 1918 Aug. 6. Mount Wilson.
434. Comet Brooks. 1911 Oct. 25. Helwan.

R.A.S.
Ref. No.

435. Comet Brooks. 1911 Nov. 2. Helwan.
436. Comet Brooks. 1911 Nov. 4. Helwan.
437. Nebula N.G.C. 4361. 1912 May 14. Helwan.
438. Nebula N.G.C. 7293. 1910 Nov. 1, 2. Helwan.
439. Nebula N.G.C. 6729. Showing variability, 1913–16. Helwan.
440. Nebula in Orion, in extreme red and violet light. Lick.
441. Krueger 60. 1908 July 21; 1915 Sept. 22; 1920 July 10. Barnard, Yerkes.
442. 40-inch Refractor, Yerkes Observatory.
443. Spectra of ζ UMa. Spectroscopic binary, showing lines doubled. Yerkes Observatory.
444. Spectra of μ Orionis. Spectroscopic binary, showing variable displacement. Yerkes.
445. Sunspot. 1926 Sept. 18, 20. 9-inch and 4-inch photoheliographs. R.O., Greenwich.
446. Sun. 1928 Sept. 25, 27. 4-inch photoheliograph. R.O., Greenwich.
446A. Sun. 1928 Sept. 25. 4-inch photoheliograph. R.O., Greenwich.
446B. Sun. 1928 Sept. 27. 4-inch photoheliograph. R.O., Greenwich.
447. Sunspot. 1926 Jan. 20. 4-inch photoheliograph. R.O., Greenwich.
448. Transit of Mercury. 1914 Nov. 7. 4-inch photoheliograph. R.O., Greenwich.
449. Nebular Spectra, showing red shift increasing with distance. Humason, Mt. Wilson.

REPORT OF THE EXPEDITION FROM THE ROYAL OBSERVATORY, GREENWICH, TO OBSERVE THE TOTAL ECLIPSE OF THE SUN ON 1932 AUGUST 31.

C. R. Davidson and J. Jackson.

(Communicated by the Astronomer Royal)

The site selected for the observation of the eclipse was at Parent, a divisional point on the transcontinental railway, 252 miles N.W. of Quebec. This was the most northerly site which could be easily reached, and in fact the only place well north of the St. Lawrence Valley which could be considered. Parent is about 1400 feet above sea-level and near the watershed between the St. Lawrence and Hudson Bay. The summer temperature is not much different from that of England, only occasionally rising above 80° F.

For this expedition the Admiralty agreed to pay the expenses of one observer (Dr. Jackson), while a grant from the Royal Society through the Joint Permanent Eclipse Committee allowed a second to go officially from Greenwich (Mr. Davidson). Dr. H. Slouka of Prague indicated early that he would like to join and take part in the work of erecting the instruments and making the observations. The expedition was also fortunate in having the services of Mr. W. M. Witchell of the Royal Observatory, who went as a volunteer observer. It was originally expected that an expedition from the

Dominion Observatory, Ottawa, would also go to Parent, but this had to be abandoned for financial reasons.

The following instruments were taken :—

(1) Stigmatic grating slit spectrograph for photography and photometry in the infra-red from about λ 5000 to λ 9000. An image of the Sun was formed on the slit by a 9-inch speculum mirror of 10 feet 8 inches focus. The whole spectroscope was mounted on trunnions so that it could be rotated about the line of collimation through the slit. Near the distant trunnion was a 4-inch mirror of 40 inches focus, which returned the light as a parallel beam to a concave diffraction grating of one metre focus placed near the slit. The grating was set to give a normal spectrum near λ 7000 in the first order. By curving the film to a radius of about 20 inches good definition was secured over the range from λ 5000 to λ 9000. The scale of the spectrum was 1 mm. = 17 A.

(2) Prism slit spectrograph for the comparison of the intensities of the H and K lines of calcium with the infra-red triplet near λ 8600. The image of the Sun was formed on the slit by a 15-inch silver-on-glass mirror of 29 feet 2 inches focus. After passing through the slit the light was deflected 90° by a total reflection prism to a mirror of $6\frac{1}{2}$ inches aperture and about 40 inches focus which reflected the light in a parallel beam to a 30° prism. This was used in Littrow fashion, the second face of the prism being silvered. The beam was returned to the same $6\frac{1}{2}$-inch mirror, which focused it just above the prism.

This spectrograph also was mounted so that it could be rotated about the line of collimation through the slit. It was intended to attempt near the middle of totality a determination of the colour temperature of the corona at different distances from the Sun by placing the slit radially.

(3) Objective prism of 7 inches aperture and 45° angle in conjunction with a photographic objective of 21 feet focus. The dispersion was about 1 mm. = 4 A. near λ 3600 and 1 mm. = 35 A. near λ 6000. A panchromatic film 15 inches long was used. The focal curve was determined at Greenwich with a reflecting collimator. The absolute focus was determined by autocollimation both at Greenwich and at Parent.

(4) Coronagraph of 45 feet focus and 6 inches aperture. The lens by Alvan Clarke is the property of Mr. Worthington, who extended the loan for this eclipse.

(5) A 6-inch Simms lens (visual) of 6 feet focus to be used with a red filter and a panchromatic plate. The purpose of this instrument was to find whether the effect of fogging by the corona could be cut down sufficiently to admit of sharp star images relatively near the Sun. As there was a poor star field no attempt was made to determine the amount of the deflection of star light.

All the instruments were fed by cœlostats. The grating spectrograph had a 12-inch cœlostat. One 16-inch cœlostat was used for (3) and (5), while each of the other two had a 16-inch cœlostat. The cœlostats were covered by huts 8 feet square, taken from England, and for (1) and (2) the huts also covered the spectrographs.

The instruments were all prepared at Greenwich during the spring and early summer, and despatched by Admiralty lorry from the Observatory on July 13. The observers sailed from Southampton on July 20 by the C.P.R. steamer *Montcalm* for Quebec, the instruments going by the same steamer. From Quebec the transport was by rail, and the instruments arrived at Parent on August 1. The observers were accommodated in a private house, and a suitable site for the instruments was selected on the other side of the road. The weather throughout August was very wet. On one day it was estimated that about 6 inches of rain fell—most of it in 6 hours, while on many other days the rainfall must have exceeded 1 inch. The Sun was seen on relatively few days, but on about 4 days, including August 29, the sky was magnificent. During the first week of August the nights were very fine and the sky exceedingly transparent.

Concrete piers were erected for the four cœlostats, for the two image formers and for the 21-foot telescope. The spectrographs and driving clocks were supported on stands made for them at Greenwich, the 45-foot on packing-cases, and the 6-inch Simms on improvised stands.

The 45-foot and the 21-foot telescopes were both focused by auto-collimation, in the case of the latter the image of a carbon arc (containing calcium) was focused on a slit temporarily inserted in the camera back. We would like again to point out the necessity of sheltering completely the long horizontal tubes from direct sunlight. The focus of both instruments was known from previous eclipses, and from observations made at Greenwich. On account of the strong winds which frequently arose at Parent, the Sun screens were not erected there till the evening of August 29. The first attempt to focus the 45-foot was after dark on a day when there had been sunshine. The images showed strong astigmatism due to stratification of air in the horizontal tube, and at first the vertical and horizontal directions indicated focus at 10 cm. and 5 cm. less than had been used at previous eclipses. Even 3 hours after sunset the images were far from perfect and the focus was still several centimetres short. Early the next morning, when it was cloudy, good images were found with the focus at the position we had expected it to be from previous eclipses. With the 21-foot the focus was found during an afternoon to be 4 mm. shorter than had been determined at Greenwich and the images were poor, but the next morning the focus was found to have returned to the position found at Greenwich, and at the same time the images had greatly improved. It appears that once the focus has been determined under good conditions it should not be departed from without good reason.

On account of the inclement weather, the focusing of the slit spectroscopes would have given great trouble in the field. Fortunately we obtained, from the Canadian National Railways, the use of an unoccupied house. Here we were able to set up the two slit spectrographs and run a carbon arc (110 volts, 60 cycles). The kitchen, where running water was available, was used as a dark room. As the house could be used as a workshop and for protecting the delicate instruments—mirrors, photographic plates, etc.—it was of the greatest use in the trying weather conditions. Although it

was generally cool and the tap-water below 60° F., great trouble was experienced in the development of the infra-red plates, and ice had to be used for cooling the developer.

After a nearly perfect day on August 29, it was cloudy all day on August 30, although the Sun showed faintly through the clouds for an hour or so in the morning. On August 31 it was cloudy practically continuously. Twice in the morning we glimpsed the Sun through the clouds and checked the orientation of the 45-foot telescope—on account of the Sun's motion in declination, the camera end had to be moved 5 inches per day. After noon one or two small breaks in the clouds passed over, one of them about 5 minutes before totality. The corona was seen before second contact, but the clouds were thickening, and after mid-totality nothing was visible till near the end when prominences and corona were again faintly visible.

During the eclipse the instruments were operated as follows :—

(1) Grating spectrograph by Mr. Davidson.
(2) Prism spectrograph by Dr. Jackson.
(3) Objective prism by Mr. Witchell.
(4) 45-foot coronagraph by Dr. Slouka.
(5) Simms coronagraph by Mr. A. Frechette.

The time was counted by Father Caillier and Dr. Henry, who took alternate 10 seconds.

As the camp was only about 200 yards north and 50 yards west of a survey pillar whose position was determined astronomically in 1914, there was no difficulty in fixing our position accurately. The co-ordinates of the site were longitude 74° 36′·5, latitude 47° 55′·3, about 5 miles S.W. of the central line of the eclipse track. From the data in the *Nautical Almanac* and the *American Ephemeris*, the computed time of second contact was $20^h\ 18^m\ 7^s$ G.C.T.

A communication from the office of the *American Ephemeris*, received some time before the expedition sailed, made the time of second contact about 8 seconds later, and actually the observed contact was 3 or 4 seconds later still. There is no doubt that this was due to the form of the Moon at the point of contact. Mr. Davidson observed the crescent to hang for several seconds. Dr. Jackson, who was observing a large faint image of the Sun (82 mm. diameter) on the slit (20 mm. long) of his spectrograph, saw the crescent slowly fade out without the formation of any beads. The photographs taken with the 45-foot coronagraph show a distinct flattening of the advancing edge of the Moon, while the flash spectrum with the objective prism show a sudden end to the crescents especially on the side of greater position-angle, and no beads. The eclipse appeared to last only 99^s as compared with the computed 102^s. Perhaps half a second or a little more can be explained by the rate of the metronome, but probably most of it is to be accounted for in the 3 or 4 seconds the eclipse appeared to be late as a result of the flattening of the Moon's limb.

It might be objected that the flattening of the Moon on the photographs was not real, but due to spreading of the image of over-exposed prominences

and bright coronal matter. To check this the third plate taken with the 45-foot coronagraph was measured in the solar micrometer at Greenwich. This plate was chosen as the image was very faint and there were no bright parts near the point of second contact. Prominences between position-angles 260° and 340° produced unevenness, but outside this region there were few irregularities apart from those due to the figure of the Moon. There was a depression amounting to 2″ extending about 8° on each side of position-angle 132°—the point of second contact. It should be pointed out, however, that the photograph shows no effect of refraction which should have reduced the vertical diameter relative to the horizontal (position-angle 47°) by about 1″·5; and the effect of the Moon's motion during the exposure is not apparent.

When the eclipse was over and before the photographs were developed, it was expected that no results of value had been obtained. The grating spectrograph showed only the hydrogen and helium lines, although the exposure was made early. The weakness of the lines compared with those obtained with the objective prism indicated that the instrument had not sufficient light grasp for use in the infra-red with plates of the present sensitivity. The other slit spectrograph also gave no results, although in this case there was undoubtedly sufficient light grasp. On account of the clouds the image former (15 inches aperture and 29 feet focus) gave a rather faint image to follow on the slit. The 45-foot coronagraph gave photographs of the prominences and a little inner corona with exposures of 3^s, 10^s, 30^s and 3^s. In comparing the photographs with those taken with the same instrument at Alor Star in 1929 when the light transmitted through the clouds was approximately the same, it is remarkable that in 1932 there was no trace of scattered light inside the Moon's disc, while there was a large amount in 1929. Apparently in the earlier eclipse the clouds were high and scattered a considerable amount of light, while in the later one the clouds were low and absorbed it. The Simms coronagraph showed a typical minimum corona, but no star images.

Considering the cloudiness of the sky, the results obtained with the objective prism are remarkably good. Four films were taken as follows :—

(1) From second contact till 7 seconds afterwards.
(2) From 10^s to 45^s.
(3) From 50^s to 85^s.
(4) From 91^s to end of totality.

Very little is shown by (3) and (4) as the clouds were densest while these exposures were being made. No. (2) shows a number of lines, and in particular the coronium lines at λ 5303 and λ 6374, which are quite different in appearance. The line at λ 6374 is shown as a nearly complete ring of practically uniform intensity. On the other hand, the line at λ 5303 shows generally on the western side of the Sun, but is missing on the eastern side. It is specially strong near position-angle 280°, where also there is fairly strong continuous spectrum. At this point the corona shows strongly on the direct photographs, above a low-lying prominence. At Greenwich

flocculi were observed in this region with the spectrohelioscope, and there had previously been a sunspot there. On August 28 a bright eruptive prominence had been seen in the same place.

The photograph taken at the first flash is very good. The lines are numerous and in good focus from λ 3590 to the limit of sensitivity of the panchromatic plate, the helium line at λ 7065 being shown. It is not considered, however, that it will show any lines which have not been photographed by one or other of the past observers of flash spectra. Although heights might be deduced from the lengths of the arcs, these would be very uncertain for the lower lines, on account of the irregularity of the Moon near the point of second contact. Many high-level lines which are visible 30° from the geometrical point of contact are invisible 20° from it.

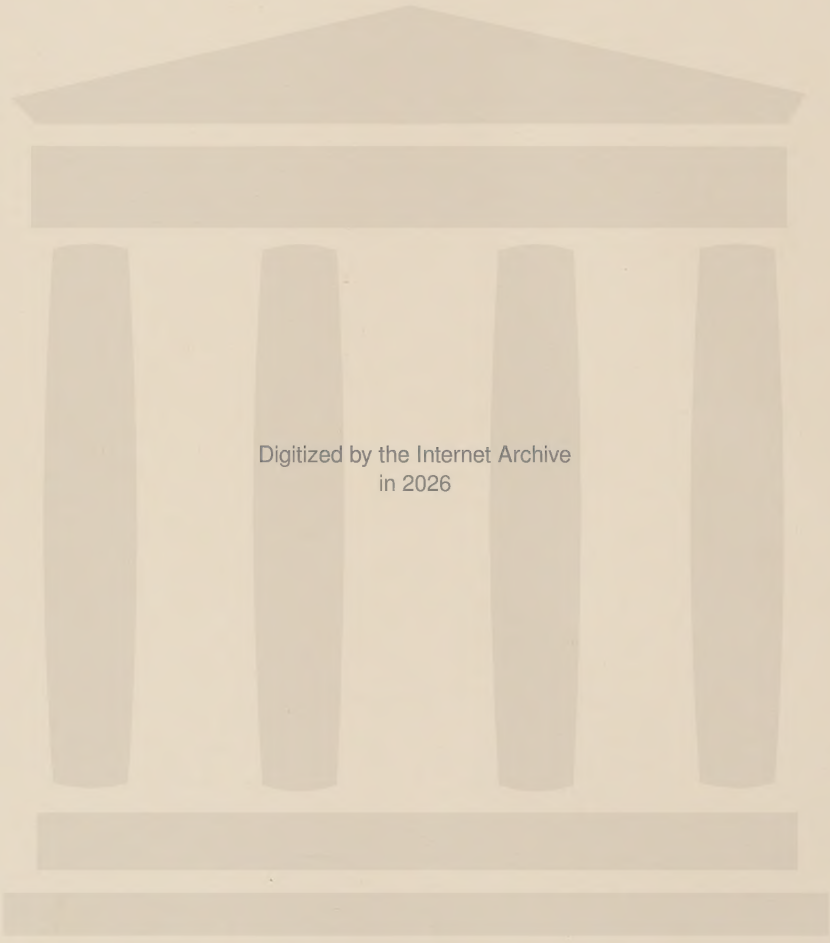
The objective prism photographs were taken to show the distribution of the various elements in the chromosphere and prominences, and for comparison with the direct photographs of the corona secured with the large coronagraph. As regards wave-length determinations and the identification of the elements in the chromosphere, the best spectra already secured with objective prisms and gratings or with slit spectrographs cannot greatly be improved on in the region covered by the panchromatic plate except with more powerful instruments under favourable conditions. The great need for the future appears to be accurate photometric intensities of the lines, especially of multiplets, and these can only be determined when a comparison spectrum can be secured with the same instrument. The comparison spectrum must be obtained from the uneclipsed Sun or an artificial source, and for this a collimator and slit are necessary.

But it is not necessary to use a narrow slit in taking the eclipse spectra. In 1927 Minnaert used a wide slit, and there appears to be no reason why the slit should not be widened still further, perhaps even to take in the whole of the Sun. It will, of course, be necessary to focus the Sun's image accurately in the plane of the slit. This method would be easier for the observer, as there would be no fear of placing the slit beyond the Sun's limb. It would also enable the conditions at various positions along the Sun's limb to be investigated, and avoid objections to methods which use a limited portion of the chromosphere.

From the photographs taken by Fowler in 1898 it is clear that with a large objective prism excellent spectra can be taken in less than a second. This has been abundantly confirmed by the spectra taken with a moving plate by Campbell, whose photographs show that most of the chromospheric lines are produced at such low levels that the effective exposure is only a second or less. Mitchell has secured excellent photographs with objective gratings in which the light grasp must be smaller. It should be possible with a prism, or prisms, of large aperture to take spectra in well under a second, and we would suggest that these should be taken through a wide slit on the cinematograph principle. By cutting down the exposure and making a number of exposures it should be possible to determine more accurately the variation of intensity with height above the photosphere. The instrument most suitable appears to be of the type used successfully

East Limb. Second contact.

Davidson and Jackson, Eclipse of 1932 Aug. 31.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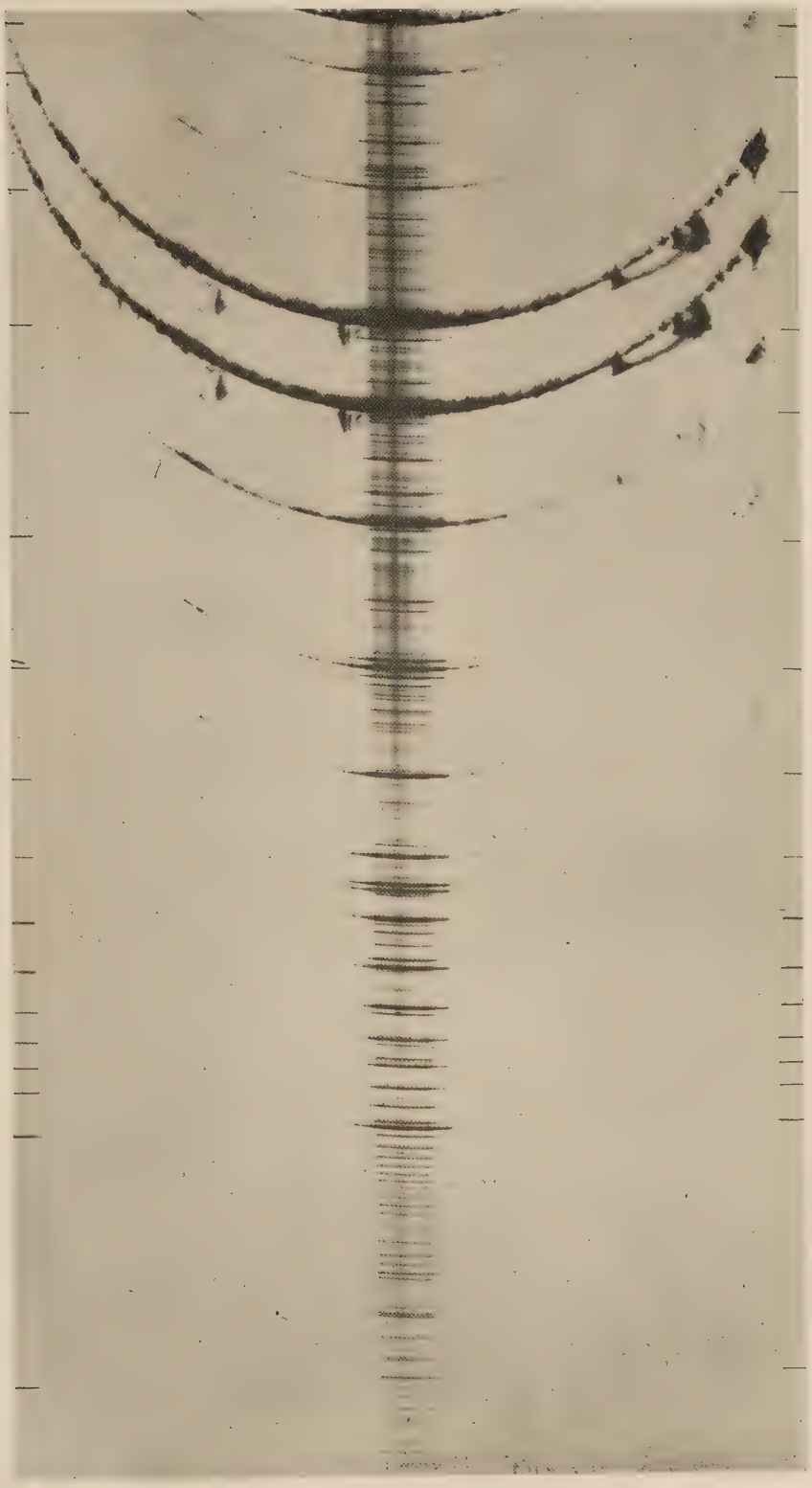
West Limb. $20^s - 30^s$.

Davidson and Jackson, Eclipse of 1932 Aug. 31.

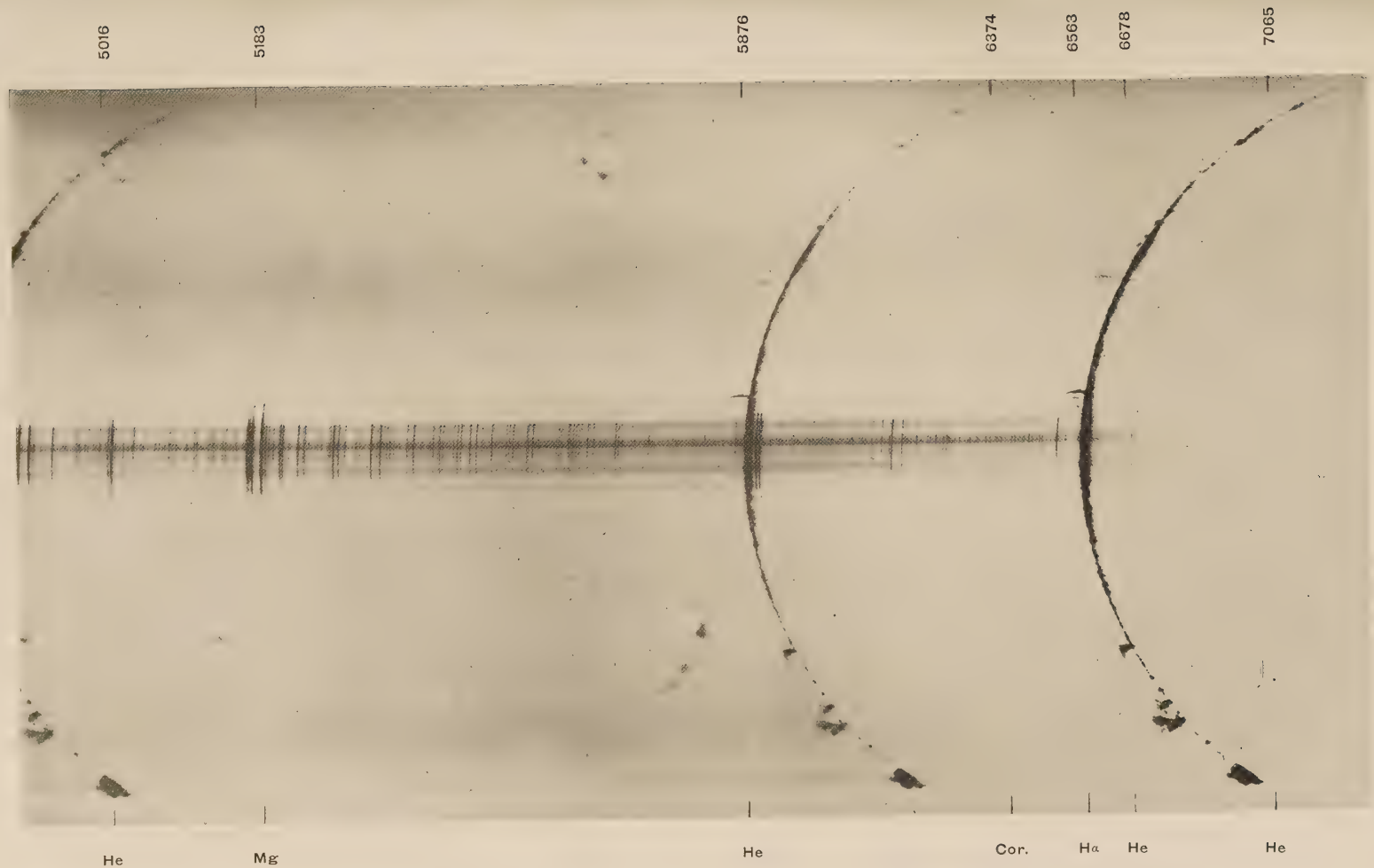
West Limb. $45^s - 75^s$. Very cloudy.

Davidson and Jackson, Eclipse of 1932 Aug. 31.



Spectrum of the Chromosphere with objective prism. Region from λ 3600 to λ 4100. Second contact.

Davidson and Jackson, Eclipse of 1932 Aug. 31.



Spectrum of the Chromosphere with objective prism. Region from λ 5000 to λ 7000. Second contact.

Davidson and Jackson, Eclipse of 1932 Aug. 31.

by Minnaert in 1927. If a long range of wave-lengths is to be studied, the image ought to be formed by a mirror. For investigation of low levels of the chromosphere, observations should be made from a position not far inside the belt of totality, so that the layers may be observable for a relatively long time.

It is a pleasure to acknowledge assistance received from the Canadian Pacific, and from the Canadian National Railways. We are indebted to Dr. R. M. Stewart, Director of the Dominion Observatory, Ottawa, who visited Parent and gave information as to the possibility of the site. He also made arrangements with the Canadian Customs authorities, who allowed all the instruments to pass through the customs unexamined. At Parent the observers were hospitably welcomed, and they received the greatest kindness and assistance from everyone with whom they came in contact.

TOTAL SOLAR ECLIPSE, 1932 AUGUST 31: REPORT OF THE EXPEDITION FROM THE SOLAR PHYSICS OBSERVATORY, CAMBRIDGE.

F. J. M. Stratton, D.S.O., M.A.

An expedition to observe the total solar eclipse of 1932 August 31 was arranged under the joint auspices of the Joint Permanent Eclipse Committee and the Solar Physics Observatory, Cambridge. The original party consisted of Professor F. J. M. Stratton, Professor J. A. Carroll, Dr. R. O. Redman and Mr. C. P. Butler. Owing to the meeting of the International Astronomical Union at Harvard early in September and to the accessibility of the belt of totality in Canada, an unusual amount of skilled volunteer help was available for the day of the eclipse itself and for a few days preceding it. It was decided that this provided an opportunity for carrying out a large programme of observation at little extra expense, save for the time involved in preparation in England. All available instruments were therefore pressed into service, and special arrangements were made to feed extra beams from auxiliary mirrors driven along with the main cœlostats. These mirrors were a source of anxiety in the preparatory stages and never very satisfactory. When possible, mirrors independently controlled should be used for eclipse work.

Dean Eve of McGill University, Montreal, very kindly offered to assist the expedition in every way, and with the kind co-operation of the authorities of the Hermitage Country Club, Magog, selected a very suitable site for the eclipse camp between two fairways of the local golf-course. The eclipse occurred in the afternoon, and there was a good view to the south-west over ground sloping down to Lake Memphremagog. By the kindness of the Canadian Minister of Defence, tents to shelter the instruments were provided without charge by the Canadian Defence Force, and when a number

of these tents were damaged badly by a line squall, which struck the camp, they were all very speedily replaced.

The programme consisted of the following main groups of experiments :—

(*a*) Spectrophotometric study of the flash and the corona.
(*b*) Spectroscopic observations in the infra-red.
(*c*) Interferometric studies of coronal and chromospheric lines.
(*d*) Polarisation effects in the corona.
(*e*) Coronal photographs.
(*f*) Rotation of the corona.
(*g*) Air temperature and humidity changes.

(a) *Spectrophotometry.* (i) *Moving-plate Camera.*—This was an objective spectrograph with a 6-inch plane grating and a 6-inch Cooke photovisual lens of 20 feet focal length ; it was fed from an auxiliary mirror mounted on the polar axis of the Cambridge 16-inch cœlostat. The slit in the focal plane was 1 mm. wide, and the plate was driven under gravity past this slit at 1·5 mm./sec. by means of a steel wire running off a drum driven by a gramophone motor. Time-marks were made on the plate by light signals controlled by chronometer contacts. The slit was horizontal, and was normal to the Sun and 9° from the points of contact. Exposure requirements being uncertain, two plates of different speeds were used for the two contacts—Imperial Eclipse Ortho 850, and Imperial Special Rapid, both backed. The effective exposure was $\frac{2}{3}$ second. The other halves of the plates were calibrated in an auxiliary spectrograph. The region of spectrum to be examined was the ordinary photographic region $H\beta - H\delta$. Focusing was done by an auxiliary collimator equipped with a concave mirror of $6\frac{1}{4}$ feet focal length. Dr. R. O. Redman was in charge of this instrument.

(ii) *The 21-foot Rowland Concave Grating.*—A beam from the 21-inch siderostat, kindly lent by Sir Francis McClean, was fed on to a Lassell $7\frac{1}{4}$-inch mirror of 85 inches focal length. An image of the Sun was formed on the slit of a collimator with a 12-inch mirror (Oliver) of 97 inches focal length. The parallel beam from this mirror fell on the Rowland concave grating and the photographic plate was placed at the principal focus, where the images being stigmatic were suitable for photometric work. The plates used were curved and backed, the spectrum reaching from λ 4900 to λ 3000. The region of the spectrum to which special attention was paid was from the head of the Balmer series λ 3646 towards the ultra-violet. The direction of dispersion was parallel to the line of contacts, and for that the instrument had to be tilted on its concrete pillars 14° from the horizontal.

Comparison spectra to be used for photometric work were to be obtained from out-of-focus images of the Sun taken immediately after the eclipse through the same optical train, but with the Lassell mirror moved to 14 feet 6 inches from the slit of the collimator and with a calibrated wedge (platinum sputtered on quartz by Professor Carroll at Aberdeen) over the slit. To provide for exposures of different length on the flash and the corona the solar comparisons were taken with and without a quartz diffuser in the beam. Further comparisons were to be made on a tungsten filament lamp calibrated

with a fixed current at the National Physical Laboratory. The somewhat heavy set of batteries required to give 16 amp. D.C. for the lamp were kindly lent by the Hart Company of Canada, and a petrol-charging set for these batteries and for running D.C. arcs was lent by the Canadian Defence Force.

Mr. Butler was in charge of the setting up of this instrument, and at the eclipse was in general control, with Mr. J. Murray at the camera and Mr. Lang at the siderostat.

(iii) *The Hills 4-prism Quartz Spectrograph.*—This instrument was to be used for the continuous spectrum at the head of the Balmer series, and to repeat with different dispersion from the Rowland grating the spectrophotometric study of the lines of the Balmer series at different levels in the Sun. The continuous spectrum of the corona in the ultra-violet was also to be studied. The calibration programme was similar to that of the Rowland concave grating. The image of the Sun on the slit of the spectroscope was formed by a 9-inch Herschel speculum mirror of 10 feet focal length, fed by a 9-inch speculum flat, kindly lent by Dr. Steavenson and the Royal Observatory, Greenwich, respectively. The flat was mounted on a base linked to and driven with the Cambridge 16-inch cœlostat.* Process films were used. The instrument was worked by Dr. W. L. Webster and Dr. C. S. Beals.

(iv) *The 20-inch Common Reflector.*—The short focal length of this instrument (45 inches), and its large light-gathering power, made it suitable for a study of radiation from an extended source. A 1-prism slit spectroscope, working at $f/3$ with two Aldis anastigmat lenses of 12-inch focus, was built for use with this telescope, which was mounted equatorially. A study of the distribution of the continuous radiation in the corona at different levels and the absorption lines at higher levels was intended. The programme of exposures (1 second, 8 seconds and 60 seconds) was chosen to link up with the programme of the Greenwich expedition, where an instrument of larger focal ratio was available. Ilford Special Panchromatic plates were used. Mr. E. G. Williams, with the help of Mr. G. Murray, was in charge of the instrument.

(b) *Spectroscopic Observations in the Infra-red.*—Two regions were especially studied for new coronal lines and for the flash spectrum. As several other expeditions nearby were known to be working on the spectral region sensitive to the mesocyanine dye (maximum 7500 A.), it was decided to allot the instrument which gave the better optical performance to the region centred at 9600 A., for which the plates (Eastman Infra-red Sensitive B) were stained with xenocyanine. These plates were not available in England for trial before the expedition left for Canada, but Professor G. H. Henderson of Dalhousie University, Halifax, kindly undertook to experiment with these plates before joining the eclipse camp, and he subsequently took charge of the whole of the infra-red work and the hyper-sensitising of the special plates.

The two instruments used were both fed with beams from the 21-inch siderostat. For the xenocyanine plates the beam was diverted by a 12-inch

* *M.N.*, 87, 677, 1927.

flat (belonging to the Hills siderostat) on to a 9-inch concave speculum mirror of 10 feet focal length, kindly lent by the Science Museum. The spectrograph used was a Hills slit spectrograph, with a 4-inch Anderson grating having a very bright first-order spectrum. An argon tube was used for a comparison spectrum. Professor Henderson, with the help of Mr. L. Lefèvre, took charge of this instrument. The beam for the other spectrograph was diverted by a 5-inch right-angled prism on to a 5-inch Rowland flat grating used objectively. The image of the flash spectrum was formed by a 6-inch Henry lens of 90 inches focal length. Professor J. Satterly took charge of this instrument.

(c) *Interferometric Studies of Coronal and Chromospheric Lines.*—The instruments from Aberdeen brought by Professor J. A. Carroll consisted of two objective interferometers as follows :—

(i) A large interferometer fed from the 16-inch Oxford cœlostat (kindly lent by Professor H. H. Plaskett), with a pyrex mirror 12 inches in diameter ; the optical train consisted of two 45° prisms 6 inches in height, followed by a Fabry-Perot étalon, and a single lens of 13·5 cm. aperture and focal length 15 cm. The étalon consisted of a pair of plates 13 cm. in diameter, silvered by cathode deposition *in vacuo* so that each plate reflected about 70 per cent. of green light falling on it near normal incidence. They were mounted in a specially designed carrier separated by $\frac{3}{16}$ of an inch, and tilted about an axis in the plane of dispersion so that the angle of incidence was between 20° and 30°. It will be seen that this instrument was of very great light-gathering power, and, used with special plates prepared by Messrs. Ilford, Ltd., and the *H* emulsions by the Eastern Kodak Co., it is thought that ample exposure would have been secured in less than half a minute. Two exposures were planned—one of 10 seconds with a fast, coarse-grained plate, and one of 50 seconds with a slow, fine-grained plate. The resolving power of the interferometer is, of course, scarcely more than could be obtained with any good spectroscope ; but naturally the use of the instrument objectively in this manner yields information which a slit spectroscope cannot give without the use of the method of the spectro-heliograph. The objective interferometer yields a picture of the corona in its own monochromatic radiation crossed by the Fabry-Perot fringes whose shape, width and intensity can be measured at all points of the image. The isolation of the monochromatic image of the corona in the green light (5303 A.) was much facilitated by the special plates employed, which had a sharp maximum of sensitivity at just this wave-length. The plates were standardised by means of comparison fringes from a high-voltage thallium arc. It is expected that full details for the construction of this instrument will be published separately in the near future. Thanks are very specially due to Mr. C. G. Fraser, of the Department of Natural Philosophy, Aberdeen, for much care and thought over the design of this instrument, and the very beautiful workmanship displayed in its construction. Professor Carroll took charge of this instrument, Captain W. M. Lindley being in control of the cœlostat.

(ii) A small interferometer fed from the same pyrex mirror consisted

of an étalon of plates 64 mm. in diameter, somewhat more heavily silvered, and separated by $\frac{3}{32}$ of an inch and with a dispersion train of two 60° light flint prisms. The camera lens was a 21-inch focus Ross Process Xpres of exceptionally fine performance. Messrs. Ross very courteously sent lenses to Professor Carroll to test, from which this one was selected. This instrument was designed for measurements of the profiles of the chromospheric lines by providing a flash spectrum of crescents in the usual manner, crossed by the interference fringes formed by the etalon (which was tilted at about 30° to the line of sight). The instrument was set to cover the whole of the visible spectrum, but the exposures were planned so that the stronger chromospheric lines only would be photographed, to avoid too great confusion with the moderate scale of dispersion employed. Exposures were also planned with this instrument on the coronal green line during the greater part of totality in the same manner as with a large interferometer. The exposures on the flash, of course, were to be made in the few seconds after second contact and preceding third contact, consisting of one of 1 second and one of about 5 seconds. Dr. J. L. Johnstone was in charge of this instrument.

All the photographic plates used in both instruments were calibrated for photometric measurements.

(d) *Polarisation Effects in the Corona.*—The double-tube camera with a large Nicol prism (kindly lent by Professor Newall) in front of one lens, as taken out to the eclipses of 1927 and 1929, was set up for a study of the distribution of polarisation in the corona. Four exposures of 10 seconds, each with the axis of the prism in positions displaced each time by 45°, were to be taken. The photometric calibration was done by a tube photometer and skylight. The instrument was in the charge of Mr. S. Stratton and Mr. G. Dodd.

The beams for the above camera came from the Cambridge 16-inch cœlostat. Another beam from the same cœlostat fed a double-image slit spectrograph designed for the study of the polarisation of the corona in different wave-lengths. The image of the Sun was formed by a Hilger lens of focal length $28\frac{1}{4}$ inches, working at $f/8{\cdot}4$. The slit of the spectrograph was placed diametrically across the Sun's image. The two polarised images were formed by a prism of fluorspar placed between the dispersing image of the spectrograph and the camera lens. The direction of separation of the images was at right angles to the dispersion and parallel to the axis of the Nicol prism of the double-tube camera in its initial position. The comparison spectra for the photometric study were taken with a calibrating spectrograph. Exposures of 15 seconds and 60 seconds were to be made for the study of the lower and higher corona. Professor Stratton took charge of this instrument.

The fluorspar prism used in this spectrograph was lent by Professor Newall, as was also a Savart prism through which Dr. Aston was to examine the sky in the immediate neighbourhood of the Sun for evidence of polarised light scattered into the coronal beam by the Earth's atmosphere, as noticed by Professor Newall at Guelma in 1905.*

* *M.N.*, **66**, 476, 1906.

(e) *Coronagraphs.*—The Greenwich 19-foot coronagraph used as at Benkoelen in 1926, with the 4-inch lens and the 8-inch Dublin cœlostat (both kindly lent by the Royal Irish Academy), was the main instrument used for direct photography of the corona. Calibrated plates for photometric purposes were taken with a tube photometer and skylight with the same exposures, namely, 2, 5 and 15 seconds. A 4-inch direct-vision prism (kindly lent by the Royal Observatory, Edinburgh) was to be placed in front of the object-glass for the flash spectrum at second and third contacts. Dr. Knox-Shaw, with the help of the Master of Gonville and Caius College, Mr. J. F. Cameron, was in charge of the instrument. A spare beam from the cœlostat was used for two cine-cameras and a stationary camera worked by Mr. G. Cohen.

In addition to Dr. Hall's two cameras, to be described below, Dr. Lockyer provided a coronagraph of 4 inches aperture and 20 inches focal length, which he worked himself. This was provided with a beam from the Oxford cœlostat, with an auxiliary flat supplied by Professor Carroll.

(f) *Rotation of the Corona.*—The Newall 4-prism spectrograph mounted on a polar heliostat was arranged for a study of the rotation of the corona. The instrument was mounted as used by Professor Newall at Pulgaon in 1898,* save that, instead of two slits being used, one slit was used and set diametrically across the Sun at the two ends of the equator. A comparison spectrum from an arc of calcium and manganese was to be put on the plate immediately after totality. The instrument was in the charge of Dr. J. L. Haughton, assisted by Mr. G. Stratton.

(g) *Air Temperature and Humidity Changes.*—Dr. Haughton carried out measures of air temperature by means of a multiple thermocouple and a potentiometer. The couple was made from twelve pairs of iron-constantan wires, soldered in series, the "cold junctions" being in glass tubes passing through the cork of a 1-quart thermos flask. The thermos was buried up to its neck in the ground and filled with crushed ice. The thermocouples were arranged radially round the flask, being parallel to the ground for about 6 inches and then bending up for about 4 inches, so that the junctions lay in a rough circle of about 15 inches in diameter and about 6 inches above the ground. A canvas sheet screened the couples from direct sunshine. This screen was fixed about 6 feet above the ground on the N.E. side and about 1 foot on the S.W. side.

The apparatus was calibrated by comparing the readings obtained on the potentiometer with those of a standard mercury thermometer mounted in the same plane as the couple junctions. Neglecting one reading which lies exactly 1° C. off the curve, and which is probably a mistaken reading of the thermometer, there was no observation more than 0°·2 C. away from the curve, and the majority were within 0°·1 C. of it. In view of the fact that the potentiometer readings were definitely unsteady when there was a breeze, oscillating over about 0·1 or 0·2 millivolts, the calibration can be considered very satisfactory. If this method is used in a future eclipse, it might be well to shelter the couple from direct draught, so as to render its reading a little more steady.

* *M.N.*, **58**, App. [55], 1898.

Readings of the potentiometer were taken at intervals of approximately 20 minutes for about 4 hours before first contact, and then more frequently. The results obtained in the neighbourhood of totality are shown in fig. 1.

Between 15^h and 18^h 30^m G.C.T. the temperature tended upwards, presumably due to the clearing of the sky. After this, as the clouds returned, the temperature fell steadily. The rate of fall did not appear to increase after first contact, until totality was approaching, when an acceleration does seem to have occurred, a sharp drop of about $0°{\cdot}5$ C. taking place during totality. It is not, however, certain that this $0°{\cdot}5$ C. fall was due to the eclipse, as departures from a smooth curve of twice this amount can be seen on other parts of the record. The total fall in temperature between the first and third contacts was of the order of $3°{\cdot}5$ C. This may be compared

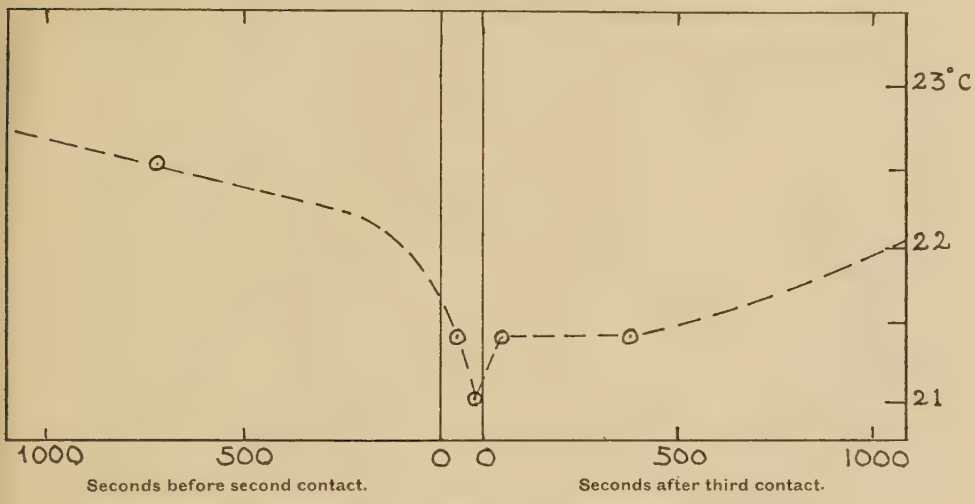


FIG. 1.—*Air Temperature during Eclipse.*

with the fall of about 1° C. observed at Penhill during the eclipse of 1927.* In this latter case, however, the fall was superimposed on a rising temperature, while at Magog the temperature was already falling.

The only safe deduction to make from these results is that during a cloudy eclipse the fall of temperature is very slight, and may readily be masked by changes in cloud conditions.

Professor A. N. Shaw carried out observations of air temperature, humidity, katathermometer readings and pressure. A drop of over 5° F. in air temperature was accompanied by an increase of relative humidity from 66·8 to 82 per cent., the absolute humidity remaining nearly constant (see fig. 2). A further drop of temperature of 6°, or a higher relative humidity at the time of first contact, might easily have led to a general deposit of dew. Professor Shaw from a study of the observations calculates that if the Sun had not been obscured by cloud the total fall of temperature would have been between 12° and 15° F. The need to protect eclipse apparatus of small thermal capacity from cooling is emphasised by these

* *M.N.*, **87**, 692, 1927.

results. The velocity of the wind was determined from katathermometer readings. The wind, which varied in direction between S.W. and S., fell from 7·0 m.p.h. at first contact to 0·74 m.p.h. a little before totality. During totality it had an average of 1·7 m.p.h., falling to 0·56 m.p.h. just after. The barometer during the time of eclipse slowly fell 0·05 inch. Miss M. Dodd helped Professor Shaw in recording the observations.

Dr. Hall's Apparatus.—Dr. Wilfred Hall joined the Magog eclipse camp,

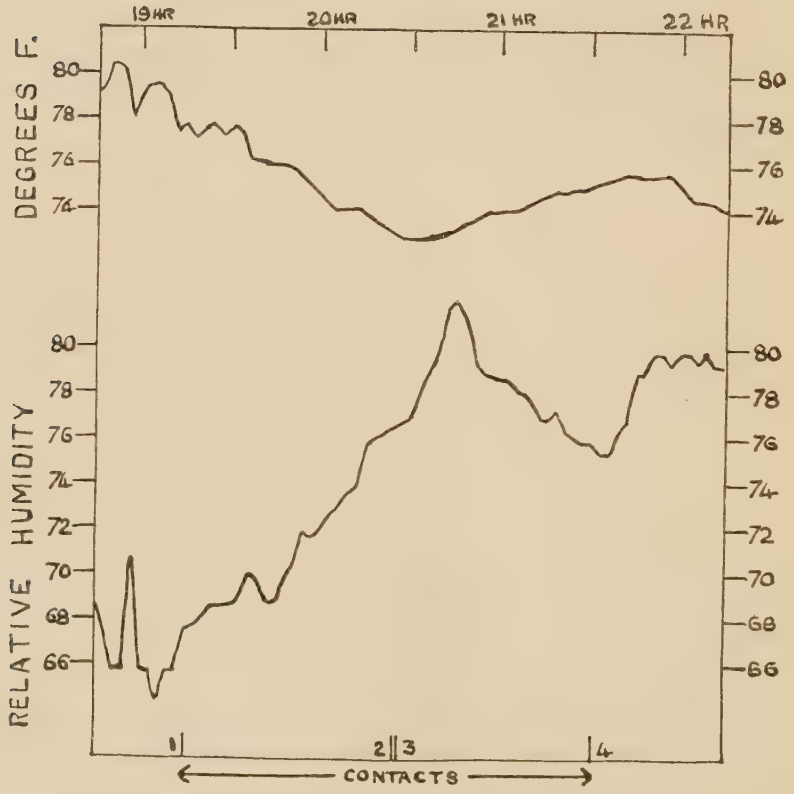


Fig. 2.—*Humidity and Temperature.*

bringing his own coronagraphs and spectrographs. His programme was arranged to fit in with the main work of the expedition.

(1) *Coronagraphs.*—These were a camera with a Cooke P.V. lens of 5 inches aperture and a focal length of 100 inches, fed by a Grubb cœlostat, and a small camera mounted equatorially, having a W.D. aeroplane lens of 19½ inches focal length working at f/5·8. With the former it was intended to photograph the corona without any screen, also by light transmitted through a green filter as well as through a cell containing a solution of æsculin. A photometric scale for comparison of densities would have been taken with the tube photometer if results had been obtained. With the small camera it was hoped to photograph the outer corona with the special object of determining the extent of any coronal extensions—a long exposure

being given with this object. Mr. A. D. Thackeray and Mr. P. M. Evans were in charge of the above coronagraphs.

(2) *Spectrographs.*—An attempt was made to obtain small-scale spectra of the corona in the ultra-violet and infra-red. For this purpose two spectrographs were arranged, one a stellar U.V. Hilger instrument mounted equatorially, the Sun's image being formed by a quartz lens (kindly lent by the Managers of the Royal Institution); the other a Bellingham & Stanley spectrograph working at f/2, an infra-red "N" plate being used. This instrument was fed by a small polar heliostat. A suitable lens of short focal length for forming the image was kindly supplied by Professor Carroll. Dr. Hall and Mr. R. Short worked these two spectrographs.

By arrangement with Dean Eve the Hills 12-inch siderostat was placed at the disposal of a party of observers from McGill University. Dr. J. S. Foster and Dr. A. V. Douglas, with a party of helpers, decided in the end, for the sake of electrical and general camp facilities, to place this instrument also at Magog. Alongside also was Professor S. A. Mitchell with a party of observers from the Leander McCormick Observatory. To him the expedition was indebted for many acts of kindness.

Weather conditions during the month preceding the eclipse were on the whole more favourable than adverse, but the afternoon of August 31, after a promising morning, was one of continuous heavy cloud. A glimpse of the crescent a minute or so before totality was all that was seen of the partial phase, and thick cloud intervened for the actual time of totality.

The debts which the expedition owes to many individuals and institutions have been indicated above. But this report cannot close without an expression of gratitude to the authorities of the Hermitage Club for the facilities provided by them; to Mr. J. Dodd (who counted the seconds during totality) and other volunteers who gave frequent help to the expedition during the weeks of preparation; to Mr. S. Stratton for making all the electric arrangements inside the camp; and to Colonel J. Waley Cohen, who once again acted as honorary secretary to the expedition, formed the advance party in pitching the camp and the rear party in striking it, arranged in advance for electric supply, dark-room facilities and accommodation for the whole party, and helped throughout in all the details of administration which need attention if a large party of observers are to carry out a heavy and elaborate programme.

TOTAL ECLIPSE OF THE SUN, 1932 AUGUST 31: EXPEDITION TO GORHAM, NEW HAMPSHIRE.

R. L. Waterfield.

The expedition to Gorham, New Hampshire, experienced complete failure as a result of clouds. However, four members of the party observed totality in a clear sky by driving for about thirty miles in a fast car.

The expedition was planned and carried out by the following:—Dr. D. T. Bonney (Baltimore), Professor D. S. Hartline (Bloomsburg, Pa.), Dr. H. K. Hartline and Dr. A. L. Patterson (University of Pennsylvania), Mr. G. A. Hobart 3rd (Paterson, N.J.), Mr. B. M. Peek, Rev. T. E. R. Phillips and myself.

The eclipse camp was two miles from Gorham, N.H., in a secluded spot on the Androscoggin River, about eleven miles from the centre of the track. Totality was computed to last about 95 seconds.

Instruments and Programme

Five groups of instruments were set up—

I. Polar axis A, carrying three infra-red coronagraphs: (*a*) Double coronagraph with two 4-inch Wray lenses of 5-foot focus. (*b*) 6-inch Cooke lens of 7-foot focus. (*c*) 4-inch Cooke Triplet working at F4·5.

From photographs taken at the Philippine eclipse in 1929 Dr. Hartline, Dr. Lloyd and myself concluded that the distribution of the infra-red light in the corona differed from that of the shorter wave-lengths (*M.N.*, **90**, 605, 1930). We hoped to confirm this with the double coronagraph, which was designed to take two simultaneous long exposures on the two halves of a single infra-red plate, one through a violet, the other through an infra-red filter. Further, photometric squares could afterwards be imprinted upon the plate *via* the same two filters. The Ilford Infra-Red plate, that does not require hypersensitizing, was to be used.

On the 6-inch and 4-inch coronagraphs the more rapid Eastman Mesocyanin plates, hypersensitized and backed the night before, were to be used. The 6-inch was for the inner corona, the rapid triplet for the outer corona. All three coronagraphs were to be exposed for 90 seconds.

II. Polar axis B. This carried two coronagraphs: (*a*) 5-inch Alvan Clark lens of 6-foot focus, and (*b*) 3-inch Beck lens working at F8. Three long, simultaneous exposures were to be given with each—10, 50 and 20 seconds. With rapid panchromatic plates and yellow filters to reduce scattering we hoped to get long coronal extension.

III. Polar axis C, carrying two infra-red objective spectrographs: (*a*) $4\frac{1}{2}$-inch replica of an echelette grating was mounted in front of an Alvan Clark lens of 4-foot focus. The replica was made by Professor R. W. Wood and lent to us by him together with the lens. A high percentage of the light was thrown into the infra-red of one order, and the combination gave an intensely brilliant spectrum. The replica was mounted on the flat surface of the lens, which had therefore to be reversed; despite this the focus plates showed excellent definition between λ 7500 and λ 9500. Dispersion was 20 angstroms per millimetre. (*b*) 2-inch Thorp replica in front of a lens of 6-inch focus. This also gave a brilliant spectrum, but the dispersion was only 100 angstroms per millimetre. Both spectrographs were set with the centre of the plate at λ 8540; one exposure was to be given on each flash, and 85 seconds exposure on the corona.

IV. 4-inch Rowland grating of 5-foot focus mounted objectively with

cœlostat. A panchromatic plate was to be used for work in the yellow and red region of the flash and coronal spectra.

V. 9·3-inch Brashear lens, 12-foot focus, mounted in the Schaeberle manner. A heavy scaffolding supported the tube and lens. At its lower end a light-tight house was built for the driving-clock, the moving plate-carrier and the observer. The mechanism for the moving plate was designed and made by Mr. Hobart : its construction and adjustment is very simple. The carriage for the plate was of aluminium. Fixed to its back were three rounded steel studs, while two others were fixed to one of its edges. The carriage resting on these five "feet" travelled by sliding on two plate-glass surfaces at right angles to each other : one a circular disc $\frac{1}{4}$ inch thick and 25 inches in diameter, the other a ledge of similar glass bolted at right angles to the surface of the first. An extremely smooth motion was obtained by lubricating the surfaces with castor oil. The glass disc was supported on a heavy wooden frame, (1) below by two stout nails on which its rim rested, and (2) behind by three large lag-bolts screwed right through the frame from behind. Adjustments for squaring-on and focusing were made with the three lag-bolts, while those for orientation of the movement of the plate were made by rotating the glass disc upon its rim. A diagonal eyepiece could be fixed to the carriage so that one was able to adjust the orientation and clock-rate by direct observation of stars. Star-trails showed that there was no serious deterioration in focus up to the edges of the 10×8-inch plates. Nine exposures were to be given, ranging from $\frac{1}{2}$ to 15 seconds.

The design of the three polar axes was simple and gave great rigidity. The sides of each axis were of $\frac{7}{8}$-inch plank, held together at either end by blocks of hard wood carrying the self-aligning ball-bearings. The camera tube was mounted *inside* the axis and swung (in declination) on rough bearings consisting of two lag-bolts. The side planks of the axis were just far enough apart so that the camera tube fitted snugly between them. For focusing, etc., the camera could be turned to any declination ; ultimately it was set to that of the eclipsed Sun and secured permanently in that position by heavy screws driven through the axis into the tube. The axis above and below the contained camera was then completely boxed-in, making the whole very rigid. The bearings of the axis (in right ascension) were supported above and below on heavy scaffolding staked deeply into the ground. The drive of the three polar axes, as also of the cœlostat, was constructed on the American method of the long arm and falling clock-weight.

The Observers

The observers in charge of the various instruments were :—
Polar axis A and Infra-Red Coronagraphs : Dr. Patterson.
Polar axis B and Extension Coronagraphs : Dr. Bonney and Mr. Hobart.
Polar axis C and Infra-Red Spectrographs : Dr. Waterfield.
Rowland Grating Spectrograph : Mr. Peek and Mr. Phillips.
12-foot Coronagraph : Dr. Hartline.

Mr. E. H. Briggs counted seconds and Mr. R. W. Seeldrayers timed the contacts. The following lent valuable assistance in the working of the various instruments: Dr. Elinor Bliss, Mrs. Bonney and Mrs. Hobart, Drs. Donal, Ellsworth, McIntosh and Shaw, and Messrs. Conant and Yankovitch.

The Eclipse

On the morning of the eclipse it was brilliantly fine; but a little after noon a thick bank of clouds, coming from the west, covered the sky, hiding the Sun for the rest of the day. About an hour before totality I suggested to Mr. G. A. Hobart jun., that he might attempt to overtake by car a patch of blue sky which was receding in the south-east. He set off with Professor Hartline, Mr. Peek, and Mr. Phillips, and succeeded in reaching a point some thirty miles east of the camp, where totality was observed in a clear gap. Mr. Hobart obtained with a moving-picture camera and a hand camera some excellent pictures, showing coronal extension to one and a half solar diameters. The others noted visually a long wing to the east stretching to a distance of two and a half diameters, and two shorter diverging streamers to the west. The corona was of minimum type. With the naked eye Mr. Phillips traced the corona for several seconds after 3rd contact while Mr. Peek followed it with a hand telescope for half a minute until clouds obscured the view. To Professor Hartline and Mr. Phillips the prominences were definitely red; to Mr. Peek they appeared flame-coloured. Mr. Peek noticed a flickering effect as the shadow bands crossed his face immediately before 2nd contact.

In a second car another member of the party, Mr. Quimby, had with him a camera of 3-foot focus. At the point which he reached, the clouds covered the Sun just as Baily's Beads began to appear. At Gorham the intended programme was carried through as far as practicable, on the chance that the clouds would break.

Summary

The eclipse expedition to Gorham, N.H., got no observations owing to cloud. The programme and instrumental equipment are described, including:—

(1) A double coronagraph for the photometric comparison of the violet and infra-red corona.

(2) A new method of mounting a moving plate.

(3) A simple and rigid design for eclipse polar axes.

(4) A rapid infra-red objective spectrograph employing a large replica échelette grating.

We wish to express our sincerest thanks to Professor J. S. Ames, President of the Johns Hopkins University, for the loan of the 9·3-inch lens of the University's equatorial; to Professor R. W. Wood for lending his replica échelette and lens; to Mr. Abbott of Gorham for kindly letting us adjust our spectroscopes in one of the Power Houses of the Brown Co. of Berlin;

and to those mentioned above who helped us at our rehearsals and the eclipse. Also we wish to thank Mr. and Mrs. G. A. Hobart jun., for their many kindnesses and for giving us facilities for building our instruments in their home in Maine ; and lastly, Mrs. G. A. Hobart 3rd, who was with us in camp and helped in so many ways. Indeed, because of the kindness of these various people we shall always look back upon this unsuccessful expedition as one of the pleasantest experiences of our lives.

STELLAR PARALLAXES DETERMINED AT THE ROYAL OBSERVATORY, GREENWICH.

(Communicated by the Astronomer Royal)

The following list is a continuation of the results obtained with the 26-inch telescope, and published annually in the November number of the *Monthly Notices.*

In the column "Proper Motion" the first value given is from meridian observations ; the second is the relative proper motion derived from the photographs. The motions in both R.A. and Dec. are expressed in arc of a great circle.

In calculating the parallax the relative proper motion has been used, *not* the meridian proper motion.

In computing the P.E. an average value $\pm$ 0″·029 has been adopted as the probable error of a single plate of unit weight. This value has been adopted from the mean result for the year.

It is now the practice to make two exposures on each plate, the plate being turned through 180° after the first exposure. There now being only two images instead of three this leads to a slightly increased error for a single plate, but as the time saved is utilised in obtaining a greater number of plates, the probable error of a determination of parallax is not increased and is perhaps nearer the true probable error.

The first P.E. given is that of the parallax, the second is the P.E. of the relative proper motion.

Generally, six comparison stars of mean magnitude 11–12 have been used, of which good images are obtained with an exposure of 3 minutes. The magnitude of the parallax star has been reduced to equality by means of a rotating sector. For the brighter stars, instead of the method of local desensitisation a neutral-tinted filter absorbing 4·8 magnitudes has recently been employed in conjunction with the rotating sector. This filter is permanently mounted on one of the plate carriers employed and the carrier is reversed between the exposures to eliminate any distortion due to the filter.

The visual magnitude is taken from the *Bonn Durchmusterung*, the photographic from the *Greenwich Astrographic Catalogue.*

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type	R.A. Dec. 1900·0	Proper Motion R.A.	Proper Motion Dec.	Parallax	P.E.	No. of Plates
				h m s	″	″	″	″	
1	α UMi 88° 8	2·0	F8	1 22 34 +88° 46′	+·046	+·003	−·001	±·006	37
					+·039			±·004	
2	α UMi S.P 88° 8	2·7			+·046	+·003	+·016	±·010	26
					+·054			±·007	
					Weighted mean		+·005	±·007	
					+·044			±·005	
3	P. III 27	6·0	K2	3 15 59	−·017	+·006	+·006	±·007	29
	64° 391	7·1		+64° 14′	−·021			±·004	
4	Gr. 642	6·0	F5	3 33 56	+·156	−·060	+·018	±·009	18
	86° 51	6·0		+86° 20′	+·132			±·008	
5	P. III 160	4·9	F8	3 53 17	−·014	+·005	+·001	±·006	34
	80° 125	5·6		+80° 25′	−·024			±·005	
6	C. 580	5·0	B3	4 4 59	−·005	+·013	−·001	±·007	26
	83° 104	5·3		+83° 34′	−·015			±·006	
7	Gr. 962	7·0	Fo	5 23 56	+·044	−·111	+·011	±·007	29
	74° 249	7·1		+74° 15′	+·012			±·007	
8	24 Cam.	4·7	K5	6 45 29	+·084	−·014	+·024	±·007	30
	77° 266	5·9		+77° 6′	+·060			±·006	
9	51 H. Cep	5·0	Ma	6 53 44	−·050	−·039	−·002	±·009	21
	87° 51	6·5		+87° 12′	−·046			±·007	
10	P. VI 292	5·5	Mb	7 10 3	−·020	−·049	−·005	±·008	25
	82° 201	6·6		+82° 36′	−·004			±·005	
11	C. 1005	8·2	Go	7 11 29	−·078	−·156	+·030	±·007	27
	86° 103	8·1		+86° 35′	−·055			±·005	
12	70° 471	7·8	Ko	7 32 31	+·011	−·062	+·003	±·007	29
		8·7		+69° 55′	−·023			±·006	
13	Gr. 1374	5·2	Ko	7 48 14	−·008	−·033	+·018	±·007	25
	74° 338	6·5		+74° 11′	+·003			±·006	
14	O.A. 8475	7·5	F2	7 57 4	−·134	−·089	+·010	±·007	28
	78° 278	8·1		+77° 56′	−·120			±·005	
15	π UMa	4·7	Ko	8 31 29	−·066	+·037	+·008	±·007	28
	64° 698	5·8		+64° 41′	−·058			±·005	
16	O.A. 9309	8·5	Go	8 42 43	+·055	−·022	+·004	±·008	22
	64° 710	9·1		+64° 47′	+·073			±·007	
17	A.C. 70° 4336	...		9 34 11	−·663	−·313	+·066	±·008	27
		11·7		+70° 30′	−·653			±·004	
18	A.C. 70° 4337	...		9 34 28	−·670	−·307	+·097	±·008	27
		12·3		+70° 31′	−·653			±·004	
19	C. 1435	8·0	Go	9 44 15	−·123	−·130	+·021	±·008	25
	84° 222	8·4		+83° 57′	−·124			±·005	

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type	R.A. Dec. 1900·0	Proper Motion R.A.	Proper Motion Dec.	Parallax	P.E.	No. of Plates
				h m s	″	″	″	″	
20	63° 869	8·9 9·7	Ma	9 48 47 +63° 17′	−·300 −·309	−·520	+·068	±·009 ±·005	21
21	78° 345	9·5 11·4		10 15 4 +77° 59′	−·182 −·182	−·059	+·017	±·008 ±·005	23
22	Br. 1399 84° 234	5·0 5·8	A3	10 15 9 +84° 46′	−·109 −·114	−·036	+·036	±·008 ±·004	27
23	L.F. 1767 65° 808	7·5 7·8	Ko	10 49 45 +65° 4′	−·110 −·095	+·084	+·027	±·008 ±·004	26
24	L.F. 1770 76° 406	7·3 7·7	F5	10 51 29 +76° 15′	−·152 −·154	−·044	+·017	±·009 ±·006	23
25	Gr. 1782 81° 373	6·2 6·2	Ao	11 24 48 +81° 41′	−·162 −·150	+·047	+·015	±·009 ±·005	26
26	Br. 1634 78° 412	5·1 5·5	A5	12 7 31 +78° 10′	+·010 +·014	+·018	+·003	±·009 ±·005	27
27	80° 389	6·7 7·3	Go	12 34 8 +79° 46′	−·101 −·093	−·003	+·048	±·009 ±·006	27
28	L.F. 2523 80° 448	6·3 7·6	Ko	14 36 24 +80° 6′	−·068 −·103	+·090	+·011	±·010 ±·007	28
29	C. 2286 83° 439	9·0 9·6	Ko	15 8 26 +83° 43′	−·051 −·015	+·016	·000	±·009 ±·007	30
30	C. 2296 83° 441	8·8 9·5	Ko	15 12 18 +83° 48′	−·075 −·036	+·043	+·001	±·009 ±·005	30
31	Gr. 2263 68° 842	6·8 7·5	G5	15 33 54 +68° 8′	+·020 +·028	−·141	+·011	±·010 ±·009	25
32	π^2 UMi 80° 487	6·7 7·1	F2	15 45 7 +80° 18′	−·037 −·025	+·036	+·003	±·009 ±·006	30
33	C. 2392 81° 530	8·4 8·2	G5	15 50 45 +81° 37′	−·086 −·084	+·098	−·020	±·008 ±·004	37
34	L.F2. 231 79° 517	6·8 7·7	Ko	16 54 6 +79° 40′	+·016 −·010	+·030	−·031	±·012 ±·009	24
35	O.A. 16804 64° 1170	6·5 6·9	Ko	17 1 42 +64° 44′	+·002 −·005	+·046	+·027	±·012 ±·010	22
36	O.A. 17333 69° 925	6·5 7·9	Ko	17 32 8 +69° 40′	−·001 +·012	−·006	−·011	±·012 ±·009	26
37	O.A. 17464 72° 800	6·0 6·8	Ko	17 39 2 +72° 31′	+·021 ·000	+·014	−·013	±·012 ±·010	23
38	L. 34856 66° 1117	7·4 8·0	Ko	18 36 24 +66° 50′	+·069 +·062	−·063	+·033	±·011 ±·007	28
39	A.C. 70° 8247	... 12·7		19 1 8 +70° 30′	+·105 +·108	+·508	+·062	±·010 ±·006	31

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type	R.A. Dec. 1900·0	Proper Motion R.A.	Proper Motion Dec.	Parallax	P.E.	No. of Plates
				h m s	″	″	″	″	
40	L. 37002	7·5	Ko	19 22 5	+·002	+·018	+·016	±·009	33
	71° 949	8·0		+71° 54′	+·029			±·007	
41	L.F. 3415–6	7·8	Ko	20 11 44	−·033	−·057	−·025	±·009	26
	66° 1276	8·6		+66° 10′	−·058			±·007	
42	7 Cep	5·5	B5	21 25 50	−·006	−·008	+·003	±·010	25
	66° 1405	5·3		+66° 22′	·000			±·007	
43	69° 1229	7·9	Fo	22 9 10	−·005	−·015	−·001	±·009	25
		8·1		+69° 53′	−·021			±·005	
44	31 Cep	5·5	Fo	22 33 18	+·170	+·021	+·040	±·010	30
	72° 1049	5·6		+73° 7′	+·150			±·005	
45	34 Cep	5·0	Ko	22 47 53	+·043	+·041	−·023	±·010	31
	82° 703	6·0		+82° 37′	+·036			±·005	

Nos. 1 and 2. Two series of plates taken, at upper and lower culmination. Each series was reduced separately.

No. 5. P. III 160 = Σ 460.

No. 20. +63° 869. P.M. from *M.N.*, 80, 757.

THE RELIABILITY OF HYPOTHETICAL PARALLAXES.

W. S. Finsen.

The hypothetical parallaxes computed by H. N. Russell and C. E. Moore * have recently been criticised by D. Barbier † on theoretical grounds. The formula for the dynamical parallax may be written as

$$\pi = n\mu x,$$

where x depends on the observed arc and μ is a function of various unknown orbital elements, while $n = (\text{mass})^{-\frac{1}{3}}$ and is derived from the mass-luminosity relation. Similarly for the hypothetical parallax

$$\varpi_0 = nlx,$$

where l is a constant. Barbier points out that μ and x are correlated : to different values of x correspond special mean values of μ. We may therefore not write

$$\varpi_0 = n\bar{\mu}x,$$

where $\bar{\mu}$ is the mean of all possible values of μ. Barbier therefore concludes that Russell's parallaxes are systematically incorrect, but he is unable to supply the rigorous solution. In support of his contention he compares

* *Astro. Journ.*, 38, 89, 1928 ; and 39, 165, 1929.

† *Compt. Rend.*, 193, 709, 1931 ; and *Journ. des Observateurs*, 15, 3, 1932.

174 of Russell's parallaxes with their trigonometrical values, finding for the regression lines

$$\bar{\varpi} = 0''\cdot0059 + 0\cdot7826\varpi_0,$$
$$\bar{\varpi}_0 = 0''\cdot0038 + 0\cdot9033\varpi.$$

Correlation coefficient 0·84.

(ϖ = trigonometrical, ϖ_0 = hypothetical parallax.)

The divergence of the coefficient of ϖ_0 from unity is regarded by Barbier as confirmation of the inaccuracy pointed out by him. The object of the present paper is to show that this error does not appear in Russell's work, and to demonstrate the reliability of his parallaxes.

There are, as Russell has shown, two ways of finding the constant for use in obtaining hypothetical parallaxes, corresponding to two different conditions. These may, for brevity, be denoted by (I) and (II).

Condition I.—If stars of the same hypothetical parallax are selected, the mean of their true parallaxes should equal their hypothetical parallax.

We neglect the factor n, which is found from Russell's tables when the hypothetical parallax has been computed on the assumption of unit mass. The condition is therefore

$$\overline{\mu x} = lx.$$

But lx is now a constant, and l is a constant ; hence x is in this case constant for the group being considered. Therefore

$$l = \bar{\mu}.$$

Here the difficulty mentioned by Barbier appears : we must assign to μ the distribution appropriate to the fixed value of x. In fact, we require a table giving the value of l with argument x.

Condition II.—If stars of the same true (dynamical) parallax are selected, the mean of their hypothetical parallaxes should equal their true parallax. That is,

$$l\bar{x} = \pi.$$

But $x = \frac{\pi}{\mu}$, since $\pi = \mu x$. Therefore

$$l\overline{\left(\frac{\pi}{\mu}\right)} = \pi.$$

Remembering that π is now constant, we have

$$l = \overline{(\mu^{-1})}^{-1}.$$

Here the only assumption made is that there is no correlation between the true parallax and μ, and in finding $\overline{\mu^{-1}}$ the most general frequency distribution is to be employed.

The choice of the condition to be satisfied must depend on the way in which the hypothetical parallaxes are to be used. If stars are selected purely according to their hypothetical parallax, then we require that the mean of the true parallaxes should equal the group hypothetical parallax. This implies (I). It will not occur as frequently as might appear, for in

actual practice the selection will often be made in accordance with some criterion implying a grouping according to true parallax, and we proceed to find from the hypothetical parallaxes what the true parallax is. In this instance we require (II) to be satisfied.

If we desire that the hypothetical parallaxes should be systematically correct *for individual stars*, (II) is again required. In this particular case the observed part x has some fixed value, and therefore the refinement proposed by Barbier would be of value could it be applied.

Trigonometrical parallaxes satisfy (II). If they are used in the sense of (I), that is, if they are grouped according to trigonometric parallax, systematic corrections will be required.

We conclude that hypothetical parallaxes should conform with Condition II. This is the case with Russell's parallaxes. Here the method advocated by Barbier would lead to material improvement for individual stars only ; the only effect in general statistical use would be a reduction in the probable error, and Russell's parallaxes would not be systematically incorrect as Barbier states. This follows from the use of the mean value of the reciprocal of μ, instead of the mean of μ itself.

Suppose we have a system of parallaxes satisfying (I), and we correlate them with their *true* parallaxes. We shall then have a double frequency distribution, with two regression lines :

Mean hypothetical as a function of true : $\bar{\varpi}_0 = a + b\pi$.
Mean true as a function of hypothetical : $\bar{\pi} = c + d\varpi_0$.

As (I) is supposed to hold, $c = 0$ and $d = 1$; while $a > 0$ and $b < 1$. Hence, if these hypothetical parallaxes are compared with the true in groups, with the hypothetical as the argument, no sign of systematic deviation should be found. This would not be true if the comparison were made in the reverse sense, the difference " mean hypothetical minus true " being $a + (b - 1)\pi$. For small values the mean hypothetical parallax would be too large, and *vice versa.*

On the other hand, if (II) is supposed satisfied, as it is with Russell's parallaxes, we should find $a = 0$ and $b = 1$; while $c > 0$ and $d < 1$. If we compare the parallaxes as before, then with the true parallaxes as the argument there should be no systematic deviation. Such a deviation should be found if the grouping is made according to hypothetical, the difference " hypothetical minus mean true " being $(1 - d)\varpi_0 - c$. Small values of the hypothetical parallaxes should be smaller than the group mean of the true, and *vice versa.*

These conclusions have to be modified if the comparison is made with trigonometrical parallaxes (ϖ) instead of true. Suppose (II) to be the condition satisfied. Then the regression line $\bar{\varpi}_0 = a + b\pi$ becomes $\bar{\varpi}_0 = \alpha + \beta\varpi$. Here ϖ is definitely selected by groups, and therefore small group values of ϖ will be systematically smaller than the corresponding true parallax π, and *vice versa.* It follows that α will be > 0 and $\beta < 1$ by amounts depending on the accidental errors of the trigonometrical parallaxes, which are assumed to be systematically correct. There should be no change in the other regression line, as no special selection of ϖ occurs.

Barbier's regression lines are therefore easily explained. The first ($\bar{\varpi} = 0''{\cdot}0059 + 0{\cdot}7826\varpi_0$) is a result of comparing according to (I) a system satisfying (II). The second ($\bar{\varpi}_0 = 0''{\cdot}0038 + 0{\cdot}9033\varpi$) would be simply $\bar{\varpi}_0 = \pi$ if the comparison had been made with true parallaxes, the hypothetical parallaxes being assumed to be systematically sound. This assumption seems to be very nearly true, as will now be shown.

Nassau * has supplied the data for correction of a group of trigonometrical parallaxes. His corrections are approximately equivalent to the relation (for values of ϖ between $0''{\cdot}000$ and $0''{\cdot}100$)

$$\bar{\pi} = 0''{\cdot}002 + 0{\cdot}89\varpi.$$

The comparison of this equation with Barbier's second equation shows that there can be little reason to doubt the soundness of Russell's parallaxes. We may therefore assume that they are accurately represented by the regression lines

$$\bar{\pi} = 0''{\cdot}0059 + 0{\cdot}7826\varpi_0,$$
$$\bar{\varpi}_0 = \pi.$$

We can now easily find the " corrections " to be applied to *group* means of Russell's parallaxes. They depend on the assumption of linear regression lines, which is not quite true.

Group mean	$0''{\cdot}000$	$0''{\cdot}010$	$0''{\cdot}030$	$0''{\cdot}050$	$0''{\cdot}070$	$0''{\cdot}090$	$0''{\cdot}100$
Correction	$+0{\cdot}006$	$+0{\cdot}004$	$-0{\cdot}001$	$-0{\cdot}005$	$-0{\cdot}009$	$-0{\cdot}014$	$-0{\cdot}016$

In the introduction to his " Catalogo Generale di Parallassi Stellari," † Cecchini finds a strong systematic tendency on comparing Russell's parallaxes with trigonometrical. The comparison was made by grouping according to hypothetical parallax. Barbier quotes the following condensed table as evidence of systematic inaccuracy in the hypothetical parallaxes :

Hyp. par.	$0''{\cdot}000$	$0''{\cdot}010$	$0''{\cdot}030$	$0''{\cdot}050$	$0''{\cdot}070$	$0''{\cdot}090$	$0''{\cdot}100$
Correction	$+0{\cdot}004$	$+0{\cdot}006$	$-0{\cdot}001$	$-0{\cdot}011$	$-0{\cdot}018$	$-0{\cdot}021$	$-0{\cdot}022$

It is clear that this is merely a better determination of the "corrections" tabulated earlier from the equation of the regression line, and does not indicate a faulty system of hypothetical parallaxes. Their use is to be regarded as an attempt to realise (I) with parallaxes satisfying (II). It is to be noted, however, that they are valid only for the particular group of stars under consideration, or for material homogeneous with this group. It is therefore to be regretted that Cecchini has applied these group corrections to the hypothetical parallaxes on the supposition that they represented systematic errors. As a result the parallaxes in his catalogue are not homogeneous, the trigonometrical parallaxes conforming with Condition II, the hypothetical with Condition I.

Union Observatory, Johannesburg :
1932 *September* 14.

* *M.N.*, **88**, 585, 1928.
† *Pubbl. del R. Osservatorio Astronomico di Merate* (*Como*), No. 4, 1931.

VARIATIONS IN THE ECCENTRICITY AND SEMI-AXIS MAJOR OF THE ORBIT OF A SPECTROSCOPIC BINARY.

(Second Paper.)

M. H. H. Walters, B.A.

In a previous paper * I investigated the effect of passing stars on the orbit of a spectroscopic binary, assuming the disturbing stars to pass in hyperbolic orbits of eccentricity greater than 5 and at a distance greater than 5 times the semi-axis major of the binary. In the present paper this investigation is carried further, to include cases in which the eccentricity of the passing star's orbit lies between 1 and 5 and allowances made for a smaller distance of closest approach. We now write the first-order terms in equation (26) as

$$\frac{n_1 a_1 p^2}{k^2 m}\frac{de_1}{dt} = 2\left(\sin u_1 P_2 + \cos u_1 \frac{\partial S}{\partial v_1} P_2'\right)\frac{a_1}{p}\left(\frac{e-1}{e\cosh D - 1}\right)^3.$$

It is easy to see, by writing down a few of the new equations, that the final expression for $[e_1]$ has the same form as (56) and is in fact

$$\frac{m^2M^2}{m_2{}^3(M+m)}\frac{a_1{}^3(e-1)^3}{8192e_1}\left\{6098\left[\int_0^\infty \frac{\cos\Delta dD}{(e\cosh D-1)^2}\right]^2 + 738\left[\int_0^\infty \frac{\cos 3\Delta dD}{(e\cosh D-1)^2}\right]^2 + 17001\left[\int_0^\infty \frac{\cos\Delta\cos 2v dD}{(e\cosh D-1)^2}\right]^2 + 17001\left[\int_0^\infty \frac{\sin\Delta\sin 2v}{(e\cosh D-1)^2}\right]^2 + 1521\left[\int_0^\infty \frac{\cos 3\Delta\cos 2v dD}{(e\cosh D-1)^2}\right]^2 + 1521\left[\int_0^\infty \frac{\sin 3\Delta\sin 2v dD}{(e\cosh D-1)^2}\right]^2\right\}. \quad (1)$$

Similarly, the new expression for $\delta a_1/a_1$ is

$$\frac{27m^2M^2}{m_2{}^3(M+m)}\frac{a_1{}^2(e-1)^3}{1024}\left\{82\int_0^\infty \frac{\cos 2\Delta dD}{(e\cosh D-1)^2}\int_0^\infty \frac{2\Delta\sin 2\Delta dD}{(e\cosh D-1)^2} + 169\left[\int_0^\infty \frac{\cos 2\Delta\cos 2v dD}{(e\cosh D-1)^2}\int_0^\infty \frac{2\Delta\sin 2\Delta\cos 2v dD}{(e\cosh D-1)^2} - \int_0^\infty \frac{\sin 2\Delta\sin 2v dD}{(e\cosh D-1)^2}\int_0^\infty \frac{2\Delta\cos 2\Delta\sin 2v dD}{(e\cosh D-1)^2}\right]\right\}. \quad (2)$$

To find the new form of $2\pi\nu T g dV_0 dg$, we now have

$$V_0{}^2 = \frac{k^2M^3}{(M+m)^2}\frac{e-1}{p}$$

* *M.N.*, 92, 786, 1932.

and

$$g = p\left(\frac{e+1}{e-1}\right)^{\frac{1}{2}}.$$

So that

$$\frac{\partial(V_0, g)}{\partial(p, e)} = -\frac{kM^{\frac{3}{2}}}{M+m}\left\{\frac{1}{2}\frac{(e-1)^{\frac{1}{2}}}{p^{\frac{3}{2}}}\frac{p}{2}\frac{(-2)}{(e+1)^{\frac{1}{2}}(e-1)^{\frac{3}{2}}} + \frac{1}{2}\frac{1}{(e-1)^{\frac{1}{2}}p^{\frac{1}{2}}}\frac{(e+1)^{\frac{1}{2}}}{(e-1)^{\frac{1}{2}}}\right\}$$

$$= -\frac{kM^{\frac{3}{2}}e}{2(M+m)(e-1)(e+1)^{\frac{1}{2}}p^{\frac{1}{2}}}$$

and

$$2\pi\nu T g dg dV_0 = \frac{2\pi^2 M^{\frac{3}{2}}}{m_2^{\frac{3}{2}}(M+m)}\frac{\nu a_1^3 T}{T_1}\frac{h^{\frac{1}{2}}e}{(e-1)^{\frac{3}{2}}}dh de,$$

by which quantity we must multiply (1) and (2) before integrating these expressions over the proper ranges of h and e to obtain the final results.

Taking $T = 10^{13}$ years, $T_1 = 3{\cdot}65$ days, $a_1 = 6{\cdot}8 \times 10^{11}$ cm. $= (4{\cdot}7)^{-1} \times 10^{-6}$ parsecs (corresponding to a separation of about 0·056 a.u.), $\nu = 10$ stars per cubic parsec, $m_1 = m_2 = m/2$, the expressions outside the brackets become $12{\cdot}72 \times 10^{-7}$ and $27{\cdot}464 \times 10^{-5}$ respectively.

Let us see what value of M is given by our values of T_1 and a_1:

$$\frac{4\pi^2 a_1^3}{T_1^2} = \frac{k^2 M}{8}, \quad \text{so} \quad \frac{M}{M_\odot} = \frac{32\pi^2}{6{\cdot}66 \times 10^{-8}}\frac{(6{\cdot}8)^3 10^{33}}{(3{\cdot}65 \times 24 \times 3600)^2}\frac{1}{2 \times 10^{33}}$$

$$= 7{\cdot}5, \text{ a satisfactory order of magnitude.}$$

We have now

$$\Delta = \tfrac{1}{4}h^3(e-1)^{-\frac{3}{2}}(e \sinh D - D).$$

Denote by I_1, I_2 . . . I_6 the integrals from $D = 0$ to $D = \infty$ of the functions

$$\frac{\cos\Delta}{(e\cosh D - 1)^2}, \quad \frac{\cos\Delta\cos 2v}{(e\cosh D - 1)^2}, \quad \frac{\sin\Delta\sin 2v}{(e\cosh D - 1)^2}, \quad \frac{\Delta\sin\Delta}{(e\cosh D - 1)^2},$$

$$\frac{\Delta\sin\Delta\cos 2v}{(e\cosh D - 1)^2}, \quad \frac{\Delta\cos\Delta\sin 2v}{(e\cosh D - 1)^2}.$$

The values of these integrals are tabulated below for various values of e and h.

TABLE I

$e = 5$

h	I_1	I_2	I_3	I_4	I_5	I_6
9	0·00556	0·01508	0·01458	0·019	0·02	−0·02172
4	0·0359	0·0249	0·0213	0·0216	−0·015	0·0103
1	0·0555	0·01398	0·00353	0·00186	−0·00152	0·00355
0·1	0·0565	0·01388	0	0	0	0
0·01	0·0565	0·01388	0	0	0	0

Table II

$e=4$

h	I_1	I_2	I_3	I_4	I_5	I_6
9	0·0026	0·0214	0·02	0·02	0·0368	−0·03896
4	0·05725	0·0431	0·03825	0·0422	−0·01	0·01534
1	0·09612	0·0233	0·00656	0·00412	−0·00346	0·00656
0·i	0·0974	0·02136	0	0	0	0
0·01	0·0974	0·02136	0	0	0	0

Table III

$e=3$

h	I_1	I_2	I_3	I_4	I_5	I_6
9	0·012	0·0432	0·0289	0·035	0·0856	−0·08
4	0·10944	0·0925	0·0872	0·0928	−0·017	0·026
1	0·2035	0·0423	0·0144	0·0109	−0·0083	0·015
0·i	0·2059	0·036	0	0	0	0
0·01	0·2059	0·036	0	0	0	0

Table IV

$e=2$

h	I_1	I_2	I_3	I_4	I_5	I_6
9	0·0252	0·074	0·0754	0·112	0·272	−0·278
4	0·3044	0·322	0·3074	0·394	−0·04	0·05
1	0·702	0·109	0·0469	0·0522	−0·0423	0·0528
0·i	0·7316	0·086	0·0012	0	0	0·0012
0·01	0·7316	0·086	0	0	0	0

Table V

$e=1·25$

h	I_1	I_2	I_3	I_4	I_5	I_6
9	0·3776	0·192	0·16	0·6c0	0·944	−1·622
4	2·112	3·288	3·24	3·574	0·084	0·18
1	7·095	0·635	0·316	0·833	−0·636	0·678
0·4	7·56	0·35	...	0·175	−0·079	0·035
0·i	7·675	0·1	0·036	0·018	0	0
0·01	7·675	0·1	0	0	0	0

With the help of these tables it is possible to calculate the functions

$$H \equiv e(e-1)^{\frac{3}{2}}h^{-\frac{5}{2}}\{6098I_1^2 + 738I_1^2(3) + 17001[I_2^2 + I_3^2] + 1521[I_2^2(3) + I_3^2(3)]\}$$
$$H' \equiv e(e-1)^{\frac{3}{2}}h^{-\frac{5}{2}}\{82I_1(2)I_4(2) + 169[I_2(2)I_5(2) - I_3(2)I_6(2)]\},$$

where $I(r)$ means the value of I for r times the value of $h^{\frac{3}{2}}$ concerned. The values are tabulated in Tables VI and VII, together with the values of

$$\int_1^5 Hde \quad \text{and} \quad \int_1^5 H'de.$$

It is interesting to note that the functions I_4, I_5, I_6 are (except for a factor) the differential coefficients of I_1, I_2, I_3 with regard to h, and that δe_1 depends

only on I_1^2, I_2^2, I_3^2 while δa_1 depends only on the differential coefficients of these three quantities. There does therefore seem to be some correlation between these two quantities.

TABLE VI

H

e \ h	1	2	3	4	5	6	7	8	9	10
1·25	53518	7617·5	3076	1946	947	558·4	204·7	28·6	1·24	0·12
2	7049	1146	421·6	247·6	169·3	96·76	47·6	15·75	1·59	0·3
3	2700	486·7	183·2	93·9	57·9	35·43	18·49	6·81	1·63	0·21
4	1518	290·7	105·6	51·5	29·5	17·36	9·45	4·49	1·25	0·225
5	981	178·7	69·75	33·9	20·2	12·17	6·63	3·19	1·26	0·48
$\int_1^5 H de$	45838	6500	2677	1628	838	498·5	202·3	42·7	...	...

TABLE VII

H'

e \ h	0·63	1·26	1·89	2·53	3·15	3·78	4·41	5·04	5·67	6·30
1·25	1888	50·5	11·5	8·8	6·7	4·385	1·7	0·97	...	...
2	26·36	14·7	7·67	...	1·11	1·15	0·93	0·57	0·19	0·074
3	2·236	0·724	...	0·142	0·397	...	0·361	0·23	0·117	0·042
4	0·803	0·257	0·261	0·056	0·084	0·158	0·135	0·132	...	...
5	0·95	...	...	...	...	0·107	0·103	0·07	0·036	0·02
$\int_1^5 H' de$	...	43·4	17·35	7·88	5·6	4·1	2·38	1·435	0·825	0·23

The values of H' were not always very accurately determined, so that spaces have been left in the table, but it has in each case been easy to draw satisfactory curves through the points given. The resultant integrations yield the answers

$$\iint H\,de\,dh = 34339,$$

$$\iint H'\,de\,dh = 55{\cdot}9,$$

where the integrations have been made from $e=1$ to $e=5$ and from $h=1$ to $h=10$ (after which the values of H and H' become negligible).

The series employed in these calculations are all derivable from Ω, which is convergent for all values of h greater than unity, but we have cut off the series at the first term; if $\beta=1$, the second term is zero and the error due to our approximation amounts to the omission of a factor of the form $\left(1+\frac{q}{h^2\cosh^2 D}\right)$ from the integrands of the I's, where q is of the order 1. This error increases with decreasing h and so our values of H and H' become more and more divergent from their true values as h approaches 1, but, since $\cosh^2 D$ increases fairly rapidly, the error involved probably does not amount to much.

We find

$$e_1\delta e_1 = 0{\cdot}04368,$$

$$\frac{\delta a_1}{a_1} = 0{\cdot}01535.$$

In my previous paper I made an unfortunate slip in quoting from Jeans, due to the necessary change of notation, which involves a revision of § 11. The true value of $\frac{\partial(V_2, g)}{\partial(p, e)}$ is $-\frac{kM^{\frac{2}{3}}}{2(M+m)}\frac{1}{p^{\frac{1}{2}}e^{\frac{1}{2}}}$ to the first order, so the number of stars passing in the range e to $e+de$ and p to $p+dp$ in time T is $2\sqrt{2}\pi^2\nu a_1{}^3\frac{T}{T_1}\frac{h^{\frac{1}{2}}dhde}{e^{\frac{1}{2}}}$, and we have to integrate the expression $e^{-\frac{5}{2}}h^{-\frac{5}{2}}$ over the various ranges.

$$\int_{5\times 2^{\frac{2}{3}}}^{10\times 50^{\frac{1}{3}}}\int_{5}^{0{\cdot}01h^3}\frac{de}{e^{\frac{5}{2}}}\frac{dh}{h^{\frac{5}{2}}} = 0{\cdot}0108,$$

$$\int_{10\times 50^{\frac{1}{3}}}^{\infty}\int_{0{\cdot}0001h^3}^{0{\cdot}01h^3}\frac{de}{e^{\frac{5}{2}}}\frac{dh}{h^{\frac{5}{2}}} = 0{\cdot}0012,$$

so the integral for the first range is now 0·012.

$$\int_{5}^{5\times 2^{\frac{2}{3}}}\int_{5}^{h^3}\frac{de}{e^{\frac{5}{2}}}\frac{dh}{h^{\frac{5}{2}}} = 0{\cdot}0227,$$

$$\int_{5\times 2^{\frac{2}{3}}}^{\infty}\int_{0{\cdot}01h^3}^{h^3}\frac{de}{e^{\frac{5}{2}}}\frac{dh}{h^{\frac{5}{2}}} = 0{\cdot}012,$$

so the integral for the second range is 0·0347.

The integrals for the third and fourth ranges are 0·0048 and 0·0005 respectively ; so we have for the three terms involving Δ but not 3Δ,

$$0{\cdot}012 \times 0{\cdot}0434 + 0{\cdot}035 \times 0{\cdot}383 + 0{\cdot}0048 \times 0{\cdot}993 + 0{\cdot}0005 \times 0{\cdot}975 = 0{\cdot}01918.$$

For the value of the three other terms we have

$$0{\cdot}036 \times 0{\cdot}00394 + 0{\cdot}104 \times 0{\cdot}0364 + 0{\cdot}0144 \times 0{\cdot}1106 + 0{\cdot}0015 \times 0{\cdot}1107 = 0{\cdot}005686.$$

Finally,

$$e_1 \delta e_1 = 2{\cdot}75 \frac{\nu a_1^3 T}{T_1},$$

Similarly,

$$\delta a_1 / a_1 = 3{\cdot}2256 \frac{\nu a_1^3 T}{T_1}.$$

With the revised values for a_1 and T_1, these become 0·00026 and 0·0003, which, when added to the new values, give us finally

$$e_1 \delta e_1 = 0{\cdot}04394,$$

$$\frac{\delta a_1}{a_1} = 0{\cdot}01538.$$

We observe that both these quantities are positive and that the effect of encounters on the eccentricity of the binary is quite substantial. The effect on the dimensions of the orbit is, however, barely perceptible. The time 10^{13} years may perhaps nowadays be considered rather a too generous allowance, but if the life of the galaxy has been much less than this it seems reasonable to suppose that at the beginning, and for a comparatively long space of time, stars were much more closely packed together and that as we decrease our value of T we must increase our value of ν ; so that we should still be left with quite a large value of δe_1.

I must again express my thanks to Dr. W. M. Smart and Professor H. F. Baker for their kind advice.

DETERMINATION OF THE TRUE ANOMALY AND THE TIME OF PERIHELION PASSAGE IN HYPERBOLIC ORBITS.

Rev. M. Davidson, B.A., D.Sc.

The accurate calculation of the true anomaly and of the time of perihelion passage in the case of nearly parabolic orbits requires a disproportionate number of significant figures unless special tables are used.* Various

* H. C. Plummer, *An Introductory Treatise on Dynamical Astronomy*, p. 34 (Cambridge, 1918).

tables * have been compiled for this purpose, but as they involve the use of logarithms they are not suitable for a computing machine. The method suggested in the present paper is adapted for machine computation and will, it is hoped, enable v and T to be found with a minimum of labour. It is intended primarily for hyperbolic orbits, but is equally applicable to elliptic orbits; in each case the tables will suffice for all purposes where $e-1$ or $1-e$ is not greater than 0·01. As no hyperbolic orbits are known where e is as great as 1·01, they can be used for all orbits of this nature for values of v from 0° to 158°.

The general equation of a conic section, $r=p/(1+e\cos v)$, can be written in the form $r=p/\left(1+2e\cos^2\frac{v}{2}-e\right)$.

Hence

$$r^2=\frac{p^2}{4e^2\cos^4\frac{v}{2}}\left(1+\frac{c}{2e\cos^2\frac{v}{2}}\right)^{-2},$$

where $c=1-e$.

The area of a sector is $\int\frac{1}{2}r^2dv$; hence, expanding the expression for r^2, the area is

$$\frac{p^2}{8e^2}\int\left(\sec^4\frac{v}{2}-\frac{c}{e}\sec^6\frac{v}{2}+\frac{3}{4}\frac{c^2}{e^2}\sec^8\frac{v}{2}-\frac{1}{2}\frac{c^3}{e^3}\sec^{10}\frac{v}{2}\ \ldots\right)dv.$$

Now

$$\int\sec^4\frac{v}{2}\,dv=2\left(\tan\frac{v}{2}+\frac{1}{3}\tan^3\frac{v}{2}\right),$$

$$\int\sec^6\frac{v}{2}dv=2\left(\tan\frac{v}{2}+\frac{2}{3}\tan^3\frac{v}{2}+\frac{1}{5}\tan^5\frac{v}{2}\right),$$

and so on. Integrating each term and making the following substitutions,

$$y=\frac{c}{e},\qquad V=\tan\frac{v}{2},\qquad A=V\left(1+\frac{V^2}{3}\right),$$

$$B=V\left(\frac{V^4}{5}-1\right)+2A,\qquad C=\tfrac{3}{4}\left[V\left(\frac{V^6}{7}+1\right)+3(B-A)\right],$$

$$D=\frac{V}{2}\left(\frac{V^8}{9}-1\right)+2(A+\tfrac{4}{3}C)-3B,$$

the area of the sector is

$$\frac{p^2}{4e^2}(A-By+Cy^2-Dy^3)=\frac{Sp^2}{4e^2},\qquad(1)$$

where

$$S=A-By+Cy^2-Dy^3,\qquad(2)$$

the limits of integration being taken from perihelion to v.

* J. Bauschinger, *Tafeln zur theoretischen Astronomie*, xii, xiii (Leipzig, 1901); J. C. Watson, *Theoretical Astronomy*, tables ix and x (Philadelphia, 1867).

In the case of motion around the Sun, if t_3 and t_1 are the intervals in days from perihelion, $(t_3 - t_1)k = \frac{2 \text{ sector}}{\sqrt{p}}$, where k is the Gaussian constant. Equating this with the expression found above, and making $t_1 = 0$, so that $t_3 - T$ is the interval from perihelion,

$$t_3 - T = \frac{Sp^2}{2ke^2\sqrt{p}} = 29{\cdot}066187\frac{Sp^{\frac{3}{2}}}{e^2} \qquad (3)$$

if k be taken as 0·0172021. This value of k does not include the masses of the planets, but as it is the usual value adopted, it will be retained.

In the hyperbola $1 - e$ and hence y are negative, and all the terms inside the bracket in (1) are positive. In the case of an ellipse they are alternately positive and relative. In the parabola $y = 0$ and $p = 2q$, and (3) then becomes

$$t_3 - T = 82{\cdot}2117S = 82{\cdot}2117A = 82{\cdot}2117V\left(1 + \frac{V^2}{3}\right),$$

the well-known formula for finding the time of perihelion passage in a parabolic orbit.

When v is known and T is required, $V = \tan\frac{v}{2}$ is found, and then $A = V\left(1 + \frac{V^2}{3}\right)$. With this value of A the quantities B, C and D are taken from the tables, and S is computed from (2) and T from (3). When V exceeds 1·8, Table II is used, the values of B/A, C/A and D/C being given in this case. When $t - T$ is known and it is necessary to find v, S is first computed from (3), and then the problem is reduced to a solution of (2), where S and y are known, and V, a function of A, B, C, D, is to be found.

The solution is effected by means of Tables I or II, combined with any of the well-known tables used for finding the true anomaly in the case of parabolic motion. For the present purpose the most convenient are those by Bengt Strömgren * and Jens P. Møller.† These give $M = (t - T)q^{-\frac{3}{2}}$, from which $\tan\frac{v}{2} = V$ is found, and this can be used as the first approximation for the solution of (2). With this provisional value A is found by the method described, or from $A = 0{\cdot}012164M$, and with this value of A, Tables I or II give B, C, D. If $\Delta S = -By + Cy^2 - Dy^3$, $S - \Delta S$ is an approximate value of A, and a new value of V is now found from $M = 82{\cdot}2117(S - \Delta S)$. This is used for trial in (2), and since $\Delta S = (1 + V^2)\Delta V$, if S' is the value obtained at any trial, $\Delta V = \frac{S - S'}{1 + V^2}$. The value of V is revised until $S - S' = 0$.

If Table I is used, the provisional value of V will give the values of B, C, D without computing A.

* *Mem. B.A.A.*, **27**, pt. ii, 1927.

† *Publikationer og mindre Heddelelser fra Københavns Observatorium*, 1932.

The application of the method is shown for Comet 1926 VII (Reid). The data are as follows :—

$$e = 1{\cdot}008658, \qquad q = 0{\cdot}754732, \qquad p = 1{\cdot}515998.$$

The constants which can be used for any number of cases are:

$$29{\cdot}066187\frac{p^{\frac{3}{2}}}{e^2} = 53{\cdot}327154$$

$$y = -0{\cdot}008584, \qquad y^2 = +0{\cdot}00007368, \qquad y^3 = -0{\cdot}000000632.$$

Let $v = 123^\circ\ 15'\ 00''$, or $V = 1{\cdot}8513917$, and it is required to find $t - T$.

$$V = 1{\cdot}8513917, \qquad V^2 = 3{\cdot}427651, \qquad \tfrac{1}{3}V^2 = 1{\cdot}142550.$$

$$\begin{aligned} A &= +3{\cdot}966699 & B &= 10{\cdot}4324 \\ -By &= +0{\cdot}089552 & C &= 23{\cdot}93 \\ +Cy^2 &= +0{\cdot}001763 & D &= 54 \\ -Dy^3 &= +0{\cdot}000034 & & \\ S &= +4{\cdot}058048 & & \end{aligned}$$

$$t - T = 4{\cdot}058048 \times 53{\cdot}327154 = 216{\cdot}40415 \text{ days.}$$

When an ephemeris is required it is necessary to perform the reverse process, which is more laborious. When the first value of V is found, others can be computed very easily by utilising the ratio between the final and provisional values of V. A description of the method follows.

With the same comet, suppose the true anomaly is required 216·40415 days after perihelion. Dividing 216·40415 by 53·327154, the constant already found, the result is $S = 4{\cdot}058048$. It is required to solve the equation

$$A - By + Cy^2 - Dy^3 = 4{\cdot}058048.$$

Since $q^{\frac{3}{2}} = 0{\cdot}655676$, and $(t - T)q^{-\frac{3}{2}} = 330{\cdot}0473$, the first value of V is 1·862 from Møller's tables. Referring to Table I, provisional values of B, C, D are taken from the nearest value of V. These are 10·5803, 24·41 and 55.

$$\begin{aligned} -By &= +\ 0{\cdot}090821 \\ +Cy^2 &= \quad 0{\cdot}001798 \\ -Dy^3 &= +\ 0{\cdot}000035 \\ \Delta S &= +\ 0{\cdot}093654 \\ S &= \quad 4{\cdot}058048 \\ \text{(Provisional) } A &= \quad 3{\cdot}964394 \end{aligned}$$

$M = 82{\cdot}2117\,A = 325{\cdot}9196$, $V = 1{\cdot}850876$ from Møller's tables.

$$\begin{aligned} A &= \quad 3{\cdot}964414 & B &= 10{\cdot}4222 \\ -By &= +0{\cdot}089464 & C &= 23{\cdot}90 \\ +Cy^2 &= +0{\cdot}001760 & D &= 5{\cdot}4 \\ -Dy^3 &= +0{\cdot}000034 & & \\ S' &= \quad 4{\cdot}055672 & & \\ \Delta S &= +0{\cdot}002376 & & \\ \Delta V &= +0{\cdot}0005368 & & \\ V &= 1{\cdot}8514. & & \end{aligned}$$

$$\begin{aligned} A &= +3{\cdot}966735 & B &= 10{\cdot}4325 \\ -By &= +0{\cdot}089553 & C &= 23{\cdot}92 \\ +Cy^2 &= +0{\cdot}001763 & D &= 54 \\ -Dy^3 &= +0{\cdot}000034 \\ S &= +4{\cdot}058085 \\ \Delta S &= -0{\cdot}000037 \\ \Delta V &= -0{\cdot}0000084 \\ V &= \quad 1{\cdot}851392. \end{aligned}$$

If V be required 10 days later, $t-T=226{\cdot}40415$, and $V=1{\cdot}878645$, first approximation. In the previous case the ratio between the final and preliminary V is 0·9943, so it may be assumed that the true value of V is near $1{\cdot}878645\times 0{\cdot}9943=1{\cdot}867937$. In the present case $S=4{\cdot}133056$, and Table II is used as V exceeds 1·86.

$V=1{\cdot}867937$, $V^2=3{\cdot}489189$.

$$\begin{aligned} A &= \quad 4{\cdot}040465 & \frac{B}{A} &= 2{\cdot}6640, & B &= 10{\cdot}764 \\ -By &= +0{\cdot}092398 \\ +Cy^2 &= +0{\cdot}001845 & \frac{C}{A} &= 6{\cdot}20, & C &= 25{\cdot}05 \\ -Dy^3 &= +0{\cdot}000035 \\ S' &= +4{\cdot}134743 & \frac{D}{C} &= 2{\cdot}2, & D &= 55 \\ \Delta S &= -0{\cdot}001687 \\ \Delta V &= -0{\cdot}000376 \\ V &= \quad 1{\cdot}867561 \end{aligned}$$

This is sufficiently accurate for an ephemeris, and the process can be repeated for other values of $t-T$.

If it be assumed that the average comet is not visible beyond 1·5 unit from the Earth or 2·5 from the Sun, the equation $\frac{p}{r}=1+e\cos v$ gives $p=5\cos^2\frac{v}{2}$, since e is close to 1. Hence $V^2=\frac{5-p}{p}$ gives a maximum value for V. If, in addition, $p=1{\cdot}5$ be assumed as an average value of p, $V^2=2{\cdot}33$, or $V=1{\cdot}53$. In this case the constant $29{\cdot}066187\frac{p^{\frac{3}{2}}}{e^2}=53$, and since this is multiplied by S to give $t-T$, an error of a unit in the sixth decimal place in S will involve an error of 5 units in the fifth decimal place in $t-T$, that is, about 5 seconds. Table I gives B to five significant figures, and since y is less than 0·009, the error in the sixth decimal place in By should not exceed 1. The same remark applies to C and D. If p is small, say 0·149, as with Comet Ryves, V^2 may be as great as $\frac{5-0{\cdot}15}{0{\cdot}15}=32$, or $V=5{\cdot}6$. In this case Table II gives B to the sixth significant figure, except when V is less than 4, when only five significant figures are given. The constant $29{\cdot}066187\frac{p^{\frac{3}{2}}}{e^2}$ is now only 1·67, so that an error of 3 units in the fifth decimal

place in *By* will imply an error of only 5 seconds. The tables should suffice for all cases with a possible error of only a few seconds. It may be useful to point out that they can be used for elliptic orbits where $1-e$ is much greater than 0·01, provided v is not very high. Table I shows that if $V=0{\cdot}983$, or v is nearly 90°, an error of about one unit only in the sixth decimal place would occur in *By*, if $y=0{\cdot}1$.

TABLE I

V	*A*	*B*			*C*		*D*
0·010	0·0	0·0000			0·00		...
			1003			7	
0·100	0·1	0·1003		23	0·07		...
			1023			8	
0·197	0·2	0·2026		38	0·15		...
			1061			9	
0·292	0·3	0·3087		53	0·24		...
			1114			9	
0·381	0·4	0·4201		67	0·33		...
			1181			11	
0·466	0·5	0·5382		76	0·44		...
			1257			11	
0·546	0·6	0·6639		84	0·55		...
			1341			14	
0·620	0·7	0·7980		89	0·69		1
			1430			15	
0·690	0·8	0·9410		94	0·84		1
			1524			18	
0·756	0·9	1·0934		96	1·02		1
			1620			19	
0·818	1·0	1·2554		98	1·21		1
			1718			22	
0·876	1·1	1·4272		99	1·43		1
			1817			25	
0·931	1·2	1·6089		100	1·68		1
			1917			28	
0·983	1·3	1·8006		99	1·96		2
			2016			30	
1·033	1·4	2·0022		101	2·26		2
			2117			33	
1·080	1·5	2·2139		99	2·59		3
			2216			38	
1·125	1·6	2·4355		100	2·97		3
			2316			40	
1·168	1·7	2·6671		98	3·37		4
			2414			44	
1·210	1·8	2·9085		99	3·81		5
			2513			47	
1·249	1·9	3·1598		97	4·28		6
			2610			51	
1·288	2·0	3·4208		96	4·79		7
			2706			54	
1·325	2·1	3·6915		97	5·34		8
			2803			59	
1·360	2·2	3·9718		95	5·93		9
			2898			63	
1·395	2·3	4·2616		96	6·56		10
			2994			67	
1·428	2·4	4·5610		93	7·23		11
			3087			72	
1·461	2·5	4·8697		94	7·95		13
			3181			76	
1·492	2·6	5·1878		92	8·71		14
			3273			80	
1·523	2·7	5·5151		92	9·51		16
			3365			85	
1·553	2·8	5·8516		90	10·36		18
			3455			90	
1·581	2·9	6·1971		92	11·26		20
			3547			94	
1·610	3·0	6·5518		89	12·20		23
			3636			98	
1·637	3·1	6·9154		88	13·18		25
			3724			105	
1·664	3·2	7·2878		88	14·23		27
			3812			109	
1·690	3·3	7·6690		89	15·32		30
			3901			115	
1·716	3·4	8·0591		87	16·47		33
			3988			120	

V	*A*	*B*			*C*		*D*
1·741	3·5	8·4579		86	17·67		36
			4074			123	
1·765	3·6	8·8653		86	18·90		40
			4160			129	
1·790	3·7	9·2813		85	20·19		43
			4245			135	
1·813	3·8	9·7058		85	21·54		47
			4330			141	
1·836	3·9	10·1388		85	22·95		51
			4415			146	
1·860	4·0	10·5803		85	24·41		55
			4500			152	

Table II

V	*A*	*B/A*			*C/A*		*D/C*
1·8	3·744	2·5286			5·55		2·1
			1873			85	
1·9	4·186	2·7291		132	6·52		2·3
			2005			97	
2·0	4·667	2·9428		132	7·63	12	2·5
			2137			111	
2·1	5·187	3·1699		134	8·91	14	2·7
			2271			128	
2·2	5·749	3·4101		131	10·36	17	3·0
			2402			145	
2·3	6·356	3·6635		132	12·00	19	3·3
			2534			164	
2·4	7·008	3·9300		131	13·86	22	3·5
			2665			186	
2·5	7·709	4·2095		130	15·95	23	3·7
			2795			209	
2·6	8·459	4·5019		129	18·28	24	4·0
			2924			233	
2·7	9·261	4·8072		129	20·89	28	4·3
			3053			261	
2·8	10·117	5·1254		129	23·78	28	4·5
			3182			289	
2·9	11·030	5·4563		127	26·98	31	4·8
			3309			320	
3·0	12·000	5·8000		128	30·51	33	5·1
			3437			353	
3·1	13·030	6·1563		126	34·40	36	5·4
			3563			389	
3·2	14·122	6·5253		127	38·67	37	5·8
			3690			427	
3·3	15·279	6·9068		125	43·33	39	6·1
			4815			466	
3·4	16·502	7·3008		125	48·43	44	6·5
			3940			510	
3·5	17·792	7·7074		126	53·98	45	6·8
			4066			555	
3·6	19·152	8·1264		124	60·01	48	7·2
			4190			603	
3·7	20·585	8·5579		125	66·55	51	7·6
			4315			654	
3·8	22·091	9·0016		122	73·62	53	8·0
			4437			707	
3·9	23·673	9·4578		125	81·26	57	8·4
			4562			764	
4·0	25·334	9·9263		123	89·49	59	8·8
			4685			823	
4·1	27·075	10·4071		123	98·35	63	9·2
			4808			886	
4·2	28·897	10·9003		124	107·86	65	9·6
			4932			951	
4·3	30·803	11·4057		122	118·07	70	10·02
			5054			1021	
4·4	32·795	11·9234		123	128·99	71	10·50
			5177			1092	
4·5	34·875	12·4532		121	140·66	75	10·98
			5298			1167	
4·6	37·047	12·9954		124	153·13	80	11·47
			5422			1247	
4·7	39·308	13·5497		121	166·42	82	11·94
			5543			1329	
4·8	41·665	14·1162		122	180·57	86	12·43
			5665			1415	
4·9	44·118	14·6948		121	195·62	90	12·89
			5786			1505	
5·0	46·667	15·2857		123	211·60	93	13·41
			5909			1598	
5·1	49·318	15·8887		121	228·54	96	13·93
			6030			1694	
5·2	52·070	16·5039		122	246·50	102	14·51
			6152			1796	
				122		106	

NUMERICAL INTEGRATION OF EMDEN'S POLYTROPIC EQUATION OF INDEX THREE.

N. Fairclough, M.Sc.

(Communicated by E. A. Milne)

1. A previous paper * has been published on the integration of the equation

$$\frac{1}{\xi^2}\cdot\frac{d}{d\xi}\left(\xi^2\cdot\frac{d\theta}{d\xi}\right)=-\theta^3$$

with the initial conditions:

at $\xi=1$, $\theta=0$, $d\theta/d\xi=$ a given negative number.

Two further tables are now communicated.

We use ω_3 to denote the value of $-\xi^2\cdot\dfrac{d\theta}{d\xi}$ at the point $\theta=0$, $\xi=\xi_0$.

TABLE VIII

(Collapsed Solution) $(\omega_3 > \omega_3{}^0)$

$\xi_0=1$; $\omega_3=8{\cdot}073108$

ξ	θ	$-(d\theta/d\xi)$	$-(\xi^2\,.\,d\theta/d\xi)$
1	0	8·073108	8·073108
·95	0·424893	8·944321	8·072250
·90	0·896680	9·949149	8·058811
·85	1·421770	11·06935	7·997607
·80	2·004290	12·22621	7·824777
·75	2·642062	13·23815	7·446461
·70	3·320210	13·77154	6·748056
·65	4·002714	13·31553	5·625810
·60	4·624678	11·24653	4·048750
·55	5·091666	7·058431	2·135175
·525	5·232633	4·130154	1·138374
·50	5·294061	·708375	·177094
·49525	5·295773	·999999 × 10^{-2}	·24527 × 10^{-2}
·495183	5·295773	0	0
·49	5·293764	− ·777663	− ·186717
·47	5·247217	−3·909382	− ·863582

* *M.N.*, 91, 55, 1930.

TABLE IX

(*Centrally-condensed Solution*) ($\omega_3 < \omega_3{}^0$)

$\xi_0 = 1$; $\omega_3 = 1{\cdot}50$

ξ	θ	$-(d\theta/d\xi)$	$-(\xi^2 . d\theta/d\xi)$
1	0	1·5	1·5
·95	·078947	1·662044	1·499995
·90	·166665	1·851739	1·499908
·80	·374911	2·341235	1·498391
·70	·641934	3·042837	1·490991
·60	·994531	4·078385	1·468219
·50	1·475383	5·650972	1·412743
·45	1·784078	6·737388	1·364321
·40	2·153722	8·100372	1·296060
·35	2·600011	9·818005	1·202706
·30	3·143100	11·99215	1·079293
·25	3·809071	14·76619	·922887
·225	4·198785	16·44987	·832775
·20	4·633773	18·40057	·736023
·175	5·121864	20·72149	·634596
·15	5·674579	23·61941	·531437
·125	6·311286	27·55421	·430535
·10	7·070003	33·68057	·336806
·08	7·821175	42·28503	·270624
·06	8·818036	59·76341	·215148
·04	10·39974	107·7524	·172404
·02	14·23443	357·6279	·143051
·01	21·06334	1327·503	·132750
·005	34·05446	5122·405	·128060
·003	50·96775	13991·56	·125924
·001	133·663	122756·4	·122756

2. The following tables give the values of the associated functions u, v, z, w, defined by

$$z = \xi^{\frac{2}{n-1}}\theta(\xi), \qquad w = -\xi^{\frac{n+1}{n-1}}\theta'(\xi)$$

$$u = -\frac{\xi\theta^n}{\theta'}, \qquad v = -\frac{\xi\theta'}{\theta},$$

so that

$$u = \frac{z^n}{w}, \qquad v = \frac{w}{z}.$$

These tables, IA to IXA, are constructed to give values of the associated functions for those values of ξ chosen in the corresponding Tables I to IX.

TABLE Ia

(*Emden's Solution*) $(\omega_3 = \omega_3{}^0)$

$\xi_0 = 1$; $\omega_3{}^0 = 2{\cdot}01813$

ξ	$z(=\xi\theta)$	$w(=-\xi^2\theta')$	$u\left(=-\frac{\xi\theta^3}{\theta'}\right)$	$v\left(=-\frac{\xi\theta'}{\theta}\right)$
0	0	0	3	0
·036248	·24742	·00511	2·96404	·020653
·072496	·47992	·03871	2·85551	·080659
·108744	·68514	·11958	2·68955	·17453
·144993	·85505	·25220	2·47873	·29495
·173991	·963084	·38985	2·29137	·40479
·202990	1·046416	·54692	2·09502	·52266
·231988	1·106432	·71415	1·89664	·64545
·260987	1·145304	·88297	1·70144	·77095
·289985	1·16564	1·04604	1·51407	·89740
·326234	1·16908	1·23429	1·29453	1·05578
·362482	1·15273	1·39982	1·09422	1·21436
·398730	1·12101	1·54044	·91450	1·37415
·434978	1·07760	1·65644	·75545	1·53716
·471226	1·02544	1·74984	·61622	1·70643
·507474	·96684	1·82334	·49567	1·88588
·543723	·90360	1·87996	·39245	2·08052
·579971	·83704	1·92265	·30503	2·29696
·616219	·76819	1·95411	·23198	2·54379
·652467	·69773	1·97671	·17183	2·83308
·688715	·62624	1·99246	·12326	3·18162
·724963	·55405	2·00306	$·84909 \times 10^{-1}$	3·61530
·761212	·48143	2·00988	$·55516 \times 10^{-1}$	4·17486
·797460	·40854	2·01401	$·33857 \times 10^{-1}$	4·92978
·833708	·33551	2·01632	$·18731 \times 10^{-1}$	6·00967
·869956	·26244	2·01748	$·89594 \times 10^{-2}$	7·68739
·906204	·18931	2·01796	$·33622 \times 10^{-2}$	10·6594
·942452	·11616	2·01811	$·77655 \times 10^{-3}$	17·3743
·978701	·042998	2·01813	$·39390 \times 10^{-4}$	46·9360
·999725	·0005516	2·01813	$·83162 \times 10^{-10}$	3658·68
1·0	0	2·01813	0	∞

Table IIa

(*Collapsed Solution*) $(\omega_3 > \omega_3{}^0)$

$\xi_0 = 1$; $\omega_3 = 2{\cdot}11920$

ξ	$z(=\xi\theta)$	$w(=-\xi^2\theta')$	$u\left(=-\frac{\xi\theta^3}{\theta'}\right)$	$v\left(=-\frac{\xi\theta'}{\theta}\right)$
1	0	2·119200	0	∞
·954939	·095494	2·119190	$\cdot410920 \times 10^{-3}$	22·19189
·913762	·182752	2·119059	$\cdot288036 \times 10^{-2}$	11·59524
·875985	·262796	2·118575	$\cdot856663 \times 10^{-2}$	8·061689
·841196	·336478	2·117466	$\cdot179910 \times 10^{-1}$	6·293023
·779220	·467532	2·112329	$\cdot483807 \times 10^{-1}$	4·518042
·725569	·580455	2·101904	$\cdot930450 \times 10^{-1}$	3·621131
·678546	·678546	2·085037	·149839	3·072801
·627161	·783951	2·053972	·234570	2·620026
·582268	·873402	2·011459	·331231	2·303016
·542492	·949361	1·957668	·437074	2·062090
·506806	1·013612	1·893120	·550094	1·867697
·468290	1·077067	1·802487	·693196	1·673514
·433512	1·127131	1·698854	·842883	1·507237
·401687	1·164892	1·583774	·998080	1·359585
·372202	1·191046	1·458840	1·158186	1·224839
·344559	1·205957	1·325664	1·323005	1·099263
·318338	1·209684	1·185870	1·492723	·980314
·293166	1·201981	1·041100	1·668015	·866154
·268687	1·182223	·893032	1·850253	·755384
·244531	1·149296	·743405	2·042066	·646835
·220263	1·101315	·594045	2·248615	·539396
·195290	1·035037	·446898	2·481185	·431770
·168623	·944289	·304009	2·769670	·321945
·138047	·814477	·167145	3·232539	·205218
·111751	·681681	·078237	4·04885	·114771
·091678	·568404	·029719	6·17926	$\cdot522850 \times 10^{-1}$
·079525	·495441	·008280	14·6874	$\cdot167124 \times 10^{-1}$
·073725	·459594	·000031	3131·57	$\cdot67451 \times 10^{-4}$
·073701	·459445	0	∞	0
·07	·436253	− ·004629	− 17·93612	$-\cdot106108 \times 10^{-1}$
·06	·372465	− ·014831	− 3·484076	$-\cdot398185 \times 10^{-1}$
·04	·240865	− ·026596	− ·525417	− ·110419

TABLE IIIA

(Centrally-condensed Solution) $(\omega_3 < \omega_3{}^0)$

$\xi_0 = 1$; $\omega_3 = 1{\cdot}90$

ξ	$z(=\xi\theta)$	$w(=-\xi^2\theta')$	$u\left(=-\frac{\xi\theta^3}{\theta'}\right)$	$v\left(=-\frac{\xi\theta'}{\theta}\right)$
1·00	0	1·90	0	∞
·95	·095000	1·899989	$·451253 \times 10^{-3}$	19·99989
·90	·189995	1·899813	$·361009 \times 10^{-2}$	9·99926
·80	·379855	1·896730	$·288967 \times 10^{-1}$	4·993298
·70	·568687	1·881714	$·977387 \times 10^{-1}$	3·308874
·60	·753338	1·835720	·232897	2·436781
·50	·925138	1·725141	·458981	1·864740
·40	1·063210	1·500172	·801154	1·410984
·35	1·107057	1·328334	1·021415	1·199879
·30	1·122975	1·112891	1·272499	·991021
·25	1·099358	·860325	1·544380	·782571
·20	1·022586	·590704	1·810212	·577657
·15	·879403	·339829	2·001258	·386432
·10	·661750	·151478	1·913077	·228905
·08	·554043	·101225	1·680132	·182702
·06	·435749	$·662851 \times 10^{-1}$	1·248229	·152118
·04	·308309	$·454077 \times 10^{-1}$	·645398	·147280
·02	·173690	$·358257 \times 10^{-1}$	·146261	·206263
·01	·104286	$·340057 \times 10^{-1}$	$·333524 \times 10^{-1}$	·326081
·006	$·761183 \times 10^{-1}$	$·336365 \times 10^{-1}$	$·131116 \times 10^{-1}$	·441898
·002	$·488995 \times 10^{-1}$	$·333943 \times 10^{-1}$	$·350139 \times 10^{-2}$	·682917
·001	$·412780 \times 10^{-1}$	$·333263 \times 10^{-1}$	$·211042 \times 10^{-2}$	·807363
·0005	$·374430 \times 10^{-1}$	$·332850 \times 10^{-1}$	$·157712 \times 10^{-2}$	·888951
·0003	$·358110 \times 10^{-1}$	$·332601 \times 10^{-1}$	$·138078 \times 10^{-2}$	·928768
·0001	$·353150 \times 10^{-1}$	$·332149 \times 10^{-1}$	$·132601 \times 10^{-2}$	·940532

Table IVa

(Centrally-condensed Solution) $(\omega_3 < \omega_3{}^0)$

$\xi_0 = 1$; $\omega_3 = \cdot 201828$

ξ	$z(=\xi\theta)$	$w(=-\xi^2\theta')$	$u\left(=-\frac{\xi\theta^3}{\theta'}\right)$	$v\left(=-\frac{\xi\theta'}{\theta}\right)$
1·0	0	·201828	0	∞
·9	$\cdot 201828 \times 10^{-1}$	·201828	$\cdot 407345 \times 10^{-4}$	9·999990
·8	$\cdot 403654 \times 10^{-1}$	·201824	$\cdot 325878 \times 10^{-3}$	4·999922
·7	$\cdot 605467 \times 10^{-1}$	·201806	$\cdot 109986 \times 10^{-2}$	3·333056
·6	$\cdot 807231 \times 10^{-1}$	·201750	$\cdot 260723 \times 10^{-2}$	2·499286
·5	·100883	·201610	$\cdot 509270 \times 10^{-2}$	1·998445
·4	·120999	·201300	$\cdot 880050 \times 10^{-2}$	1·663641
·3	·140995	·200642	$\cdot 139697 \times 10^{-1}$	1·423043
·2	·160642	·199225	$\cdot 208082 \times 10^{-1}$	1·240177
·1	·179031	·195736	$\cdot 293165 \times 10^{-1}$	1·093311
·08	·182236	·194419	$\cdot 311291 \times 10^{-1}$	1·066851
·06	·185050	·192634	$\cdot 328955 \times 10^{-1}$	1·040981
·04	·187116	·190014	$\cdot 344786 \times 10^{-1}$	1·015486
·03	·187593	·188121	$\cdot 350927 \times 10^{-1}$	1·002809
·02	·187321	·185443	$\cdot 354444 \times 10^{-1}$	·989975
·01	·185145	·180951	$\cdot 350730 \times 10^{-1}$	·977350
·006	·182777	·177768	$\cdot 343490 \times 10^{-1}$	·972593
·002	·177017	·171320	$\cdot 323768 \times 10^{-1}$	·967818
·001	·173137	·167596	$\cdot 309676 \times 10^{-1}$	·967995
·0005	·169402	·164110	$\cdot 296224 \times 10^{-1}$	·968761

TABLE Va

(*Collapsed Solution*) $(\omega_3 > \omega_3^0)$

$\xi_0 = 1$; $\omega_3 = 20{\cdot}18277$

ξ	$z(=\xi\theta)$	$w(=-\xi^2\theta')$	$u\left(=-\frac{\xi\theta^3}{\theta'}\right)$	$v\left(=-\frac{\xi\theta'}{\theta}\right)$
1·00	0	20·18277	0	∞
·95	1·009025	20·16937	·50935 × 10^{-1}	19·98896
·90	2·013612	19·96010	·40904	9·91258
·85	2·989234	19·02192	1·40419	6·36348
·80	3·866675	16·50376	3·50292	4·26820
·75	4·513654	11·62648	7·90927	2·57585
·70	4·755412	4·565959	23·5523	·96016
·675	4·679012	·720536	142·170	·15399
·67	4·646884	− ·033499	−2995·39	− ·72089 × 10^{-2}
·65	4·463139	− 2·911576	− 30·5347	− ·65236
·60	3·660155	− 8·44760	− 5·80451	− 2·30799
·55	2·518679	−11·12338	− 1·43642	− 4·41635
·50	1·236429	−11·82101	− ·15990	− 9·56061
·45	− ·073186	−11·86635	·33035 × 10^{-4}	162·1392
·40	−1·381723	−11·78129	·22391	8·52652
·35	−2·625048	−10·56086	1·71282	4·02311
·30	−3·451451	− 5·838247	7·04243	1·69153
·275	−3·493482	− 2·108106	20·2247	·60344
·25	−3·191754	1·543199	− 21·0701	− ·48350
·20	−1·762568	5·332283	− 1·02689	− 3·02529
·15	·082734	5·674747	·99794 × 10^{-4}	68·5903
·10	1·877238	4·846763	1·36492	2·58186
·08	2·264805	2·700314	4·30208	1·19229
·06	1·957682	− ·277896	− 26·9988	− ·14195
·04	·915126	− 1·617361	− ·47384	− 1·76736
·03	·268510	− 1·690027	− ·11455 × 10^{-1}	− 6·29409
·02	− ·385255	− 1·686953	·33895 × 10^{-1}	4·37880
·01	− ·963452	− 1·372724	·65149	1·42480
·005	− ·944671	− ·644729	1·30757	·68249
·001	− ·331236	− ·128831	·28210	·38894
·0005	− ·225620	− ·114203	·10057	·50617
·0001	− ·148986	− ·106165	·31149 × 10^{-1}	·71259

Table VIa

(*Centrally-condensed Solution*) ($\omega_3 < \omega_3^0$)

$\xi_0 = 1$; $\omega_3 = 1{\cdot}009139$

ξ	$z(=\xi\theta)$	$w(=-\xi^2\theta')$	$u\left(=-\frac{\xi\theta^3}{\theta'}\right)$	$v\left(=-\frac{\xi\theta'}{\theta}\right)$
1·00	0	1·009139	0	∞
·95	$\cdot504574 \times 10^{-1}$	1·009137	$\cdot127298 \times 10^{-3}$	19·99980
·90	·100913	1·009111	$\cdot101837 \times 10^{-2}$	9·999772
·80	·201806	1·008648	$\cdot814815 \times 10^{-2}$	4·998122
·70	·302546	1·006392	$\cdot275175 \times 10^{-1}$	3·326407
·60	·402656	·999419	$\cdot653212 \times 10^{-1}$	2·482067
·50	·500794	·982218	·127870	1·961322
·40	·593513	·944913	·221258	1·592067
·35	·635513	·913766	·280892	1·437839
·30	·672173	·870438	·348903	1·294963
·25	·700219	·811156	·423250	1·158432
·20	·714212	·731619	·497963	1·024371
·15	·704815	·627821	·557686	·890760
·10	·655117	·498464	·564055	·760878
·08	·617628	·440800	·534490	·713698
·06	·565255	·381102	·473906	·674213
·04	·492447	·320967	·372064	·651780
·02	·388586	·261925	·224019	·674046
·01	·316539	·231925	·136752	·732691
·005	·269659	·214739	$\cdot913136 \times 10^{-1}$	·796334
·003	·245876	·206049	$\cdot721404 \times 10^{-1}$	·838020
·001	·219196	·193125	$\cdot545328 \times 10^{-1}$	·881062
·0005	·204951	·186602	$\cdot461351 \times 10^{-1}$	·910473

TABLE VIIa

(*Collapsed Solution*) $(\omega_3 > \omega_3^0)$

$\xi_0 = 1$; $\omega_3 = 4{\cdot}03626$

ξ	$z(=\xi\theta)$	$w(=-\xi^2\theta')$	$u\left(=-\frac{\xi\theta^3}{\theta'}\right)$	$v\left(=-\frac{\xi\theta'}{\theta}\right)$
1·00	0	4·036260	0	∞
·95	·201812	4·036153	·203646 × 10^{-2}	19·99953
·90	·403589	4·034473	·162941 × 10^{-1}	9·996497
·85	·605131	4·026802	·550286 × 10^{-1}	6·654429
·80	·805847	4·004982	·130665	4·969902
·75	1·004422	3·956337	·256127	3·938920
·70	1·198341	3·863032	·445465	3·223650
·65	1·383262	3·702031	·714946	2·676305
·60	1·552232	3·446633	1·085112	2·220437
·55	1·694893	3·070945	1·585457	1·811882
·50	1·797009	2·558960	2·267707	1·424011
·45	1·840982	1·918401	3·252437	1·042053
·40	1·808231	1·194832	4·948284	·660774
·35	1·683998	·474940	10·0551	·282031
·314	1·534710	·019228	187·994	·125288 × 10^{-1}
·312321	1·526555	0	∞	0
·3	1·463603	− ·134831	− 23·25308	− ·921226 × 10^{-1}
·28	1·350289	− ·327475	− 7·517995	− ·242522
·25	1·157346	− ·552313	− 2·806757	− ·477224
·22	·942153	− ·701420	− 1·192302	− ·744486

TABLE VIIIa

(*Collapsed Solution*) $(\omega_3 > \omega_3^0)$

$\xi_0 = 1$; $\omega_3 = 8{\cdot}073108$

ξ	$z(=\xi\theta)$	$w(=-\xi^2\theta')$	$u\left(=-\frac{\xi\theta^3}{\theta'}\right)$	$v\left(=-\frac{\xi\theta'}{\theta}\right)$
1	0	8·073108	0	∞
·95	·403648	8·072250	·814731 × 10^{-2}	19·99823
·90	·807012	8·058811	·652183 × 10^{-1}	9·985986
·85	1·208504	7·997607	·220691	6·617773
·80	1·603432	7·824777	·526841	4·880019
·75	1·981547	7·446461	1·044872	3·757903
·70	2·324147	6·748056	1·860424	2·903455
·65	2·601764	5·625810	3·130534	2·162306
·60	2·774807	4·048750	5·276884	1·459110
·55	2·800417	2·135175	10·28571	·762449
·525	2·747132	1·138374	17·79730	·414386
·50	2·647031	·177094	104·7306	·669028 × 10^{-1}
·49525	2·622732	·24527 × 10^{-2}	7355·587	·935170 × 10^{-3}
·495183	2·622377	0	∞	0
·49	2·593944	− ·186717	− 93·4756	− ·719818 × 10^{-1}
·47	2·466192	− ·863582	− 17·3691	− ·350168

TABLE IXa

(*Centrally-condensed Solution*) ($\omega_3 < \omega_3^0$)

$\xi_0 = 1$; $\omega_3 = 1{\cdot}5$

ξ	$z(=\xi\theta)$	$w(=-\xi^2\theta')$	$u\left(=-\frac{\xi\theta^3}{\theta'}\right)$	$v\left(=-\frac{\xi\theta'}{\theta}\right)$
1·00	0	1·500000	0	∞
·95	·075000	1·499995	·281251 × 10^{-3}	19·99994
·90	·149998	1·499908	·225005 × 10^{-2}	9·999515
·80	·299929	1·498391	·180065 × 10^{-1}	4·995825
·70	·449354	1·490991	·608543 × 10^{-1}	3·318075
·60	·596719	1·468219	·144717	2·460488
·50	·737691	1·412743	·284159	1·915087
·45	·802835	1·364321	·379282	1·699379
·40	·861489	1·296060	·493315	1·504442
·35	·910004	1·202706	·626571	1·321649
·30	·942930	1·079293	·776781	1·144617
·25	·952268	·922887	·935683	·969146
·225	·944727	·832775	1·012490	·881498
·20	·926755	·736023	1·081441	·794194
·175	·896326	·634596	1·134752	·707997
·15	·851187	·531437	1·160441	·624348
·125	·788911	·430535	1·140449	·545733
·10	·707000	·336806	1·049251	·476387
·08	·625694	·270624	·905147	·432518
·06	·529082	·215148	·688385	·406644
·04	·415989	·172404	·417542	·414443
·02	·284689	·143051	·161294	·502484
·01	·210633	·132750	·703957 × 10^{-1}	·630243
·005	·170272	·128060	·385494 × 10^{-1}	·752090
·003	·152903	·125924	·283884 × 10^{-1}	·823554
·001	·133663	·122756	·194531 × 10^{-1}	·918402

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THE RADIATIVE EQUILIBRIUM OF A PLANETARY NEBULA

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The purpose of the present paper is the consideration of the radiative equilibrium of planetary nebulæ. Recent investigations on the excitation of nebular radiation have shown that the nebular material cannot be treated simply as a "grey body," and that therefore the interaction of radiation of each wave-length with nebular material should be the subject of detailed investigation. It might seem that the introduction of radiation of many wave-lengths into the problem would so enormously increase the difficulties that it would be hopeless to try to solve it. But we shall find that the problem allows many simplifications, which are connected with the physical properties of the model and cannot disturb the adequacy of it.

The nebula is supposed to consist of hydrogen, for which all atomic constants, including the transition probabilities, are known. But, with slight modifications, the method developed here is applicable to nebulæ composed of other substances. It is certain that hydrogen is one of the most important constituents of gaseous nebulæ, and our results, corrected for the existence of other substances, will be applicable also to the real planetary nebulæ.

A method of the reduction of the spherical problem to a plane problem, proposed by Professor Milne,* is adopted in the present paper.

According to the theory of nebular luminosity developed by Zanstra, all or almost all quanta emitted from the central star having frequencies greater than ν_0 (the frequency of the limit of the Lyman series) are absorbed by the hydrogen atoms in the nebula. As was shown by Zanstra, the nebula, in the place of each absorbed quantum having a frequency greater than ν_0, creates an L_α-quantum, where L_α is the first line of the Lyman series of hydrogen. For the sake of completeness we will repeat here Zanstra's line of argument. For simplicity we will call the radiation beyond the head of the Lyman series briefly "ultra-violet radiation," and the corresponding quanta "ultra-violet quanta."

We shall consider the various transformations of an ultra-violet quantum, which is emitted from the surface of the central star. It will be absorbed by the nebular envelope, and this absorption will be accompanied by the ionisation of a hydrogen atom. After an interval of time the freed electron is captured again by a proton. There are two possibilities by this capture : I, the electron jumps immediately into the deepest level $1S$ (first level) ; and II, the electron jumps into one of the excited levels. In case I a

* Milne, *Zs. für Astrophysik*, 1, 98, 1930.

new ultra-violet quantum is emitted and the initial state is restored. In case II the electron makes a chain of transitions, the last link of which will be a transition into the first level. The dilution of radiation is so great and the density of matter is so low, that the interruption of these transitions is very improbable. The last transition into the normal state is accompanied by emission of a quantum of the Lyman series. There are again two possibilities: (*a*) An L_α-quantum is emitted. Zanstra's theory requires that the optical thickness of the nebula in ultra-violet light is at least of the order of unity. But the coefficient of the line-absorption in the Lyman series is some thousand times larger than the absorption coefficient beyond the head of the series. The emitted L_α-quantum, therefore, will be absorbed by a hydrogen atom in the normal state. This atom passes into the second level, and then, owing to the absence of external perturbations during its short lifetime, turns back to the first level, emitting again an L_α-quantum. Thus the L_α-quantum remains unchanged, and we may say that it suffers only the scattering processes. These processes may be repeated many times until the quantum reaches the outer boundary of nebula and flies away. (*b*) A quantum of some other line of the Lyman series is emitted. For simplicity we assume that it is an L_β-quantum. In this line the optical thickness of the nebula is also very large and the emitted quanta will be absorbed. This absorption is accompanied by the transition of an atom from the first to the third level. The atom in the third level has two possibilities: either it makes the transition of the type $3 \to 2 \to 1$, emitting the quanta H_α and L_α successively, or it passes immediately into the first level, emitting again L_β. In the first case the final product is an L_α-quantum. Its further fate is described above. In the second case the quantum L_β will be absorbed, and there thus exists a finite probability of creation of the quantum L_α. After many absorptions and re-emissions the probability of survival of an L_β-quantum will be very small, and the probability of creation of an L_α-quantum will be practically equal to unity.

In this manner in both cases (*a*) and (*b*) the final product is an L_α-quantum. It is easily seen that our consideration may be generalised to the cases when the transitions are accompanied, instead of by the emission of an L_β-quantum, by the emission of a quantum L_γ, L_δ, etc. Let p_1 be the probability of the case I and $1-p$ the probability of the case II. Our considerations may be summed up in the following manner.

After the absorption of an ultra-violet quantum there is a finite probability p of the re-emission of it with the same wave-length and a finite probability $1-p$ of the re-emission of the quantum L_α. We will not take into account the intermediate stages in which the absorbed quantum may appear as L_β-, L_γ-, quanta, etc. These will have little influence on our results. The quanta L_α cannot suffer new transformations and can only be scattered.

The problem in this way is reduced to the study of two superposed fields of radiation: the field of ultra-violet quanta and the L_α-radiation field in the planetary nebulæ. We shall consider first the field of ultra-violet quanta.

The Field of Ultra-violet Quanta.—As mentioned above, we shall use the method of the reduction of the spherical problem to a plane problem, developed by Professor Milne. Let k be the absorption coefficient of the ultra-violet radiation per atom. This coefficient depends upon the wave-length. We shall take its mean value. Let, further, n be the number of H atoms in the first level in 1 cm.^{3}, r_1 and r_2 the distances of the inner and outer boundary of the nebular ring from the central star. Then the optical depth at the distance r from the central star is

$$\tau = \int_r^{r_2} nkdr. \tag{1}$$

The equation of transfer of energy of ultra-violet quanta may be written, according to Milne, in the form

$$\tfrac{1}{2}\frac{dI(\tau)}{d\tau} = I(\tau) - B(\tau), \tag{2}$$

$$\tfrac{1}{2}\frac{dI'(\tau)}{d\tau} = B(\tau) - I'(\tau), \tag{3}$$

if we use an approximation of the Schwarzschild-Schuster type.

Here $I(\tau)$ is the average intensity of the diffuse ultra-violet radiation of the nebula in the outward direction at the point τ, and $I'(\tau)$ is the average intensity of the same radiation at the same point in the inward direction. The quantity $4\pi B(\tau)d\tau$ is the amount of energy of ultra-violet quanta emitted in the layer $d\tau$ per second. The same layer absorbs the diffuse ultra-violet radiation from various parts of the nebular ring. The absorbed energy is equal to $2\pi[I(\tau)+I'(\tau)]d\tau$. Besides this, the layer absorbs the radiation of the central star. Let πS be the amount of ultra-violet energy falling on each square centimetre of the inner surface of the nebula. At the point τ this amount is reduced to $\pi S e^{-(\tau_1-\tau)}$, where

$$\tau_1 \equiv \int_{r_1}^{r_2} nkdr \tag{4}$$

is the optical thickness of the nebula. From this amount our layer absorbs $\pi S e^{-(\tau_1-\tau)}d\tau$.

Since from the quanta absorbed only the fraction p is re-emitted again as ultra-violet quanta, the equation of radiative equilibrium may be written in the form:

$$p[2\pi(I+I') + \pi S e^{-(\tau_1-\ )}] = 4\pi B(\tau),$$

or

$$p[I(\tau)+I'(\tau)+\tfrac{1}{2}Se^{-(\tau_1-\tau)}] = 2B(\tau). \tag{5}$$

Introducing the boundary conditions *

$$I'(o) = o, \qquad I(\tau_1) = I'(\tau_1), \tag{6}$$

we take account of the diffuse radiation incident on any portion of the inner

* Milne, *Zs. für Astrophysik*, **1**, 105, 1930.

face of the nebular shell and arriving from other portions of the inner face.

From the equations (2) and (3) we have

$$\tfrac{1}{2}\frac{d(I+I')}{d\tau}=I-I' \tag{7}$$

$$\tfrac{1}{2}\frac{d(I-I')}{d\tau}=I+I'-2B. \tag{8}$$

Differentiating (7) and comparing with (8) we obtain

$$\tfrac{1}{4}\frac{d^2(I+I')}{d\tau^2}=I+I'-2B. \tag{9}$$

Introducing (5) in (9) we find the following differential equation for $I+I'$:

$$\tfrac{1}{4}\frac{d^2(I+I')}{d\tau^2}=(1-p)(I+I')-\frac{p}{2}Se^{-(\tau_1-\tau)}. \tag{10}$$

The general solution of this equation is

$$I+I'=Ae^{-\lambda\tau}+Be^{\lambda\tau}+\frac{2p}{3-4p}Se^{-(\tau_1-\tau)}, \tag{11}$$

where A and B are constants of integration and $\lambda=2\sqrt{1-p}$. Introducing (11) in (5) we find :

$$B(\tau)=\frac{p}{2}\left(Ae^{-\lambda\tau}+Be^{\lambda\tau}+\frac{3}{2(3-4p)}Se^{-(\tau_1-\tau)}\right). \tag{12}$$

Introducing (11) in (7) we obtain

$$I(\tau)-I'(\tau)=-\frac{\lambda}{2}Ae^{-\lambda\tau}+\frac{\lambda}{2}Be^{\lambda\tau}+\frac{p}{3-4p}Se^{-(\tau_1-\tau)}. \tag{13}$$

Adding and subtracting (11) and (13) we find $I(\tau)$ and $I'(\tau)$:

$$I(\tau)=\tfrac{1}{2}\left(1-\frac{\lambda}{2}\right)Ae^{-\lambda\tau}+\tfrac{1}{2}\left(1+\frac{\lambda}{2}\right)Be^{\lambda\tau}+\frac{3p}{2(3-4p)}Se^{-(\tau_1-\tau)}, \tag{14}$$

$$I'(\tau)=\tfrac{1}{2}\left(1+\frac{\lambda}{2}\right)Ae^{-\lambda\tau}+\tfrac{1}{2}\left(1-\frac{\lambda}{2}\right)Be^{\lambda\tau}+\frac{p}{2(3-4p)}Se^{-(\tau_1-\tau)}. \tag{15}$$

The first of the conditions (6) we may write according to (15) in the form :

$$A\left(1+\frac{\lambda}{2}\right)+B\left(1-\frac{\lambda}{2}\right)+\frac{p}{3-4p}S=0. \tag{16}$$

The second of the conditions (6) gives us :

$$\lambda Be^{\lambda\tau_1}+\frac{2p}{3-4p}S=\lambda Ae^{-\lambda\tau_1}. \tag{17}$$

From the equations (16) and (17) we find the following values of coefficients A and B :

$$A = \frac{\left(1-\frac{\lambda}{2}\right)e^{\tau_1} - \frac{\lambda}{2}e^{\lambda\tau_1}}{\frac{\lambda}{2}\left[\left(1-\frac{\lambda}{2}\right)e^{-\lambda\tau_1} + \left(1+\frac{\lambda}{2}\right)e^{\lambda\tau_1}\right]} \frac{p}{3-4p} Se^{-\tau_1}, \tag{18}$$

$$B = -\frac{\left(1+\frac{\lambda}{2}\right)e^{\tau_1} + \frac{\lambda}{2}e^{-\lambda\tau_1}}{\frac{\lambda}{2}\left[\left(1-\frac{\lambda}{2}\right)e^{-\lambda\tau_1} + \left(1+\frac{\lambda}{2}\right)e^{\lambda\tau_1}\right]} \frac{p}{3-4p} Se^{-\tau_1}. \tag{19}$$

These values of A and B introduced in (12), (14) and (15) give us the solution for the field of the ultra-violet quanta.

To compare this solution with the case when the diffuse radiation in (5) is neglected we shall consider a numerical example. For the representation of the solution in numerical form it is necessary to know τ_1 and p. Probably the values of τ_1 are different for different nebulæ. It seems that $\tau_1 = 2$ will be an appropriate value for some nebulæ. The value of p is to be calculated from pure physics. Cillié * has computed the relative probabilities of capture of electrons by protons on the different levels. From his results we have deduced the fraction of captured electrons which pass immediately from the free state to the first level, re-emitting the ultra-violet quanta. This fraction is our p. The value of p depends on the temperature of free electrons. For the different temperatures we have :

T	10,000°	20,000°	50,000°
p	0·46	0·49	0·57

Putting in (12) $\tau_1 = 2$ and $p = 0{\cdot}5$, we have the following numerical solution for $B(\tau)$:

$$B(\tau) = \tfrac{1}{4}S[\tfrac{3}{2}e^{-(\tau_1-\tau)} - 0{\cdot}032e^{-1{\cdot}4\tau} - 0{\cdot}044e^{1{\cdot}4\tau}].$$

When the diffuse radiation in (5) is neglected we find :

$$B(\tau) = \tfrac{1}{8}Se^{-(\tau_1-\tau)}.$$

If $\tau = \tau_1$, the first of these formulæ gives : $B(\tau_1) = 0{\cdot}194S$; the second, $B(\tau_1) = 0{\cdot}125S$. Thus the difference is of the same order as $B(\tau_1)$.

The Field of L_α-radiation.—For the study of the field of L_α-radiation we shall introduce the absorption-coefficient κ within the line L_α per hydrogen atom in the normal state. The optical depth for this line is defined by

$$t = \int_r^{r_2} n\kappa dr. \tag{20}$$

The ratio $\frac{\kappa}{k} = \omega$ may be supposed constant when the temperature variations within the nebula are neglected. In fact, k is a function of atomic

* Cillié, *M.N.*, **92**, 820, 1932.

constants and of the Doppler breadth of the line only. This breadth depends upon the temperature. When $\kappa/k=\omega$ is constant, the ratio t/τ is also constant, and we have

$$\frac{t}{\tau}=\frac{\kappa}{k}=\omega. \tag{21}$$

If the temperature of the nebula is of the order 10^3-10^4 degrees, the quantity ω will be also of the order 10^3-10^4. Since we have supposed that the optical thickness τ_1 of the nebula in the ultra-violet region is of the order of unity and larger, the optical thickness in the line L_α,

$$t_1=\int_{r_1}^{r_2}\kappa n dr,$$

will be of the order 10^3-10^4 or larger.

The equations of transfer of the radiation in the line L_α have the same form as (2) and (3). Let $K(t)$ be the average intensity of the diffuse L_α-radiation of nebula in the outward direction at the point t, and $K'(t)$ the average intensity of the same radiation at the same point in the inward direction. The equations of transfer are :

$$\tfrac{1}{2}\frac{dK(t)}{dt}=K(t)-C(t), \tag{22}$$

$$\tfrac{1}{2}\frac{dK'(t)}{dt}=C(t)-K'(t), \tag{23}$$

where $4\pi C(t)dt$ is the amount of energy emitted by the layer dt in the line L_α per second. This layer absorbs the diffuse L_α-radiation from the other parts of nebula. The quantity of diffuse radiation absorbed is $2\pi[K(t)+K'(t)]dt$. The number of L_α-quanta emitted by the central star is negligible, since the number of ultra-violet quanta transformed into L_α-quanta is some thousand times larger. Moreover, the quanta L_α emitted by the central star are absorbed in the deepest layers of the nebula. The number of ultra-violet quanta which are absorbed in the layer dt transformed into L_α-quanta is

$$\frac{(1-p)[2\pi(I+I')+\pi Se^{-(\tau_1-\tau)}]d\tau}{h\nu_c},$$

where ν_c is the average frequency of ultra-violet radiation.

Thus the L_α-radiation created in dt according to (5) is

$$\frac{(1-p)}{p}\frac{\nu_\alpha}{\nu_c}4\pi B(\tau)d\tau=\frac{1-p}{p}\frac{\nu_\alpha}{\nu_c}4\pi B(\tau)\frac{dt}{\omega},$$

since the amount of energy of each L_α-quantum is $h\nu_\alpha$. Hence the equation of radiative equilibrium is

$$4\pi C(t)dt=2\pi[K(t)+K'(t)]dt+4\pi\frac{1-p}{p}\frac{\nu_\alpha}{\nu_c}B(\tau)\frac{dt}{\omega},$$

or

$$C(t)=\tfrac{1}{2}[K(t)+K'(t)]+\frac{\nu_a}{\nu_c}\,\frac{1-p}{p\omega}B(\tau). \tag{24}$$

The boundary conditions are :

$$K'(0)=0\,; \qquad K'(t_1)=K(t_1). \tag{25}$$

From the equations (22) and (23) we have

$$\tfrac{1}{2}\frac{d(K+K')}{dt}=K-K', \tag{26}$$

$$\tfrac{1}{2}\frac{d(K-K')}{dt}=K+K'-2C(t). \tag{27}$$

Differentiating (26) and introducing (27) we find :

$$\tfrac{1}{4}\frac{d^2(K+K')}{dt^2}=K+K'-2C(t), \tag{28}$$

or according to (24),

$$\tfrac{1}{4}\frac{d^2(K+K')}{dt^2}=-\frac{2\nu_a}{\nu_c}\,\frac{1-p}{p\omega}B(\tau). \tag{29}$$

Writing $B(\tau)$ in the form

$$B(\tau)=\frac{p}{2}\left(Ae^{-\lambda\tau}+Be^{\lambda\tau}+De^{-(\tau_1-\tau)}\right)=\frac{p}{2}\left(Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+De^{\frac{-(t_1-t)}{\omega}}\right), \tag{30}$$

where

$$D=\frac{3}{2(3-4p)}S, \tag{31}$$

we find the following solution of equation (29) :

$$K(t)+K'(t)=a+bt-\frac{8\nu_a}{\nu_c}\,\frac{1-p}{2\lambda^2}\omega\left(Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+D\lambda^2e^{-\frac{t_1-t}{\omega}}\right),$$

where a and b are constants of integration.

Differentiating this expression we find, according to (26),

$$K(t)-K'(t)=\frac{b}{2}-\frac{4\nu_a}{\nu_c}\,\frac{1-p}{2\lambda}\left(-Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+D\lambda e^{-\frac{t_1-t}{\omega}}\right).$$

According to the definition of λ,

$$\lambda=2\sqrt{1-p}.$$

Therefore

$$K(t)+K'(t)=a+bt-\frac{\nu_a}{\nu_c}\omega\left(Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+D\lambda^2e^{-\frac{t_1-t}{\omega}}\right), \tag{32}$$

$$K(t)-K'(t)=\frac{b}{2}-\frac{\nu_a}{\nu_c}\frac{\lambda}{2}\left(-Ae^{-\frac{\lambda}{\omega}t}+Be^{\frac{\lambda}{\omega}t}+D\lambda e^{-\frac{t_1-t}{\omega}}\right). \tag{33}$$

From (32) and (33) we have

$$K(t)=\frac{a}{2}+\frac{b}{4}+\frac{b}{2}t-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1-\frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t}+B\left(1+\frac{\lambda}{2\omega}\right)e+{}^{+\frac{\lambda}{\omega}t}D\lambda^2\left(1+\frac{1}{2\omega}\right)e^{-\frac{t_1-t}{\omega}}\right] \quad (34)$$

$$K'(t)=\frac{a}{2}-\frac{b}{4}+\frac{b}{2}t-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1+\frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t}+B\left(1-\frac{\lambda}{2\omega}\right)e+{}^{\frac{\lambda}{\omega}t}D\lambda^2\left(1-\frac{1}{2\omega}\right)e^{-\frac{t_1-t}{\omega}}\right] \quad (35)$$

The first of the conditions (25) reduces to

$$\frac{a}{2}-\frac{b}{4}-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1+\frac{\lambda}{2\omega}\right)+B\left(1-\frac{\lambda}{2\omega}\right)+4D(1-p)\left(1-\frac{1}{2\omega}\right)e^{-\frac{t_1}{\omega}}\right]=0, \quad (36)$$

and the second,

$$\frac{b}{4}-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1-\frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t_1}+B\left(1+\frac{\lambda}{2\omega}\right)e^{\frac{\lambda t_1}{\omega}}+4D(1-p)\left(1+\frac{1}{2\omega}\right)\right]$$

$$=-\frac{b}{4}-\frac{\nu_a}{2\nu_c}\omega\left[A\left(1+\frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t_1}+B\left(1+\frac{\lambda}{2\omega}\right)e^{\frac{\lambda}{\omega}t_1}+4D(1-p)\left(1-\frac{1}{2\omega}\right)\right],$$

or

$$b=\frac{\nu_a}{\nu_c}[-\lambda Ae^{-\lambda\tau_1}+\lambda Be^{\lambda\tau_1}+4D(1-p)]. \quad (37)$$

Introducing (37) in (36) we find:

$$a=\frac{\nu_a}{2\nu_c}[-\lambda Ae^{-\lambda\tau_1}+\lambda Be^{\lambda\tau_1}+4D(1-p)]$$

$$+\frac{\nu_a}{2\nu_c}\omega\left[A\left(1+\frac{\lambda}{2\omega}\right)+B\left(1-\frac{\lambda}{\omega}\right)+4D(1-p)\left(1-\frac{1}{2\omega}\right)e^{-\tau_1}\right]. \quad (38)$$

The equations (34) and (35) together with (37) and (38) give the solution for the L_α-field.

The Density of Radiation in the Inner Layers of the Nebula.—We have denoted above by πS the amount of energy of the ultra-violet radiation falling on each square centimetre of the inner surface of the nebula from the star. In the absence of re-emission the mean intensity of the ultra-violet radiation in this region will be equal to $\frac{\pi S}{4\pi}=0{\cdot}25S$. In the case when re-emission is taken into account, the average intensity of the radiation increases and is equal to $\frac{1}{4}S+\frac{1}{2}(I_1+I_2)$. In the example considered above, the mean intensity of ultra-violet radiation at $\tau=\tau_1$ reaches the value $0{\cdot}39S$. We may conclude that though the re-emission increases the density of the ultra-violet radiation (which is proportional to the average intensity), this density remains of the same order as in the absence of re-emission. The state of affairs entirely changes when we consider the L_α-field. Owing to the large optical thickness of the nebula in the L_α-line, and to the fact that all L_α-quanta absorbed are re-emitted in the same frequency, the density of L_α-radiation in the inner layers of the ring is very large. In fact, in the numerical example considered above, the average intensity of L_α-radiation, $\frac{1}{2}(K_1+K_2)$ at $t=t_1$ according to (37) is

equal to 12,000S, where πS is again the energy of the whole ultra-violet radiation falling on each square centimetre of the inner surface of the nebula from the central star.* Therefore the density of L_α-radiation in the example considered is 48,000 times larger than the density of the whole diluted ultra-violet radiation of the central star. The rough estimate shows that the ultra-violet radiation of the black body at the temperatures of the order 40,000°–50,000° is about $5 \cdot 10^4$ times stronger than the same radiation within the Doppler width of the L_α-line corresponding to the temperature of the nebular matter. Thus the density of L_α-radiation in the inner layers of the nebular ring will be $48{,}000 \times 5 \cdot 10^4 = 2 \cdot 7 \times 10^9$ times larger than the density of diluted radiation of the central star within the line-width. If the dilution factor is of the order 10^{-13}, the density of diluted radiation within the width of the line will be of the order $10^{-13}\rho_\alpha$ where ρ_α is the density of the black-body radiation in the same frequency interval at the temperature of the central star. Then the density of diffuse L_α-radiation of the nebula in the inner regions of the ring will be of the order

$$27 \times 10^8 \times 10^{-13}\rho_\alpha = 0 \cdot 27 \times 10^{-3}\rho_\alpha.$$

Absorbing an L_α-quantum, the hydrogen atom from the state $1S$ passes into the state $2P$. With such a large density of radiation the excitation will be considerable. If n_2 is the number of H-atoms in the $2P$ state in one cm.^{3}, we shall have

$$n_2/n_1 = 5 \times 10^{-5} \tag{39}$$

when the temperature of the central star is 40,000°.

It is known that the absorption coefficient beyond the limit of the Lyman series is of the order $0 \cdot 5 \times 10^{-17}$. The nebular shell with optical thickness $\tau = 2$ will contain 4×10^{17} atoms of hydrogen per square centimetre. By application of (39) we find that the number of hydrogen atoms per square centimetre in the state $2P$ will be of the order 2×10^{13}. But the excitation in the outer layers is considerably lower and the value 10^{13} will be more correct. Such a great number of atoms in the state $2P$ leads to an optical thickness of the order of unity in the first two lines of the Balmer series. But the optical thickness in these lines will be sensitive to the variation of τ_0. If, for example, $\tau_0 = \frac{1}{2}$, the nebula will be transparent for each Balmer line.

* For the numerical evaluation of $\frac{1}{2}(K_1 + K_2)$ it is necessary to know the value of $\omega = \kappa/k$. According to Sugiura, the value of k beyond the limit of Lyman series is $0 \cdot 54 \times 10^{-17}$ per atom. The value of κ is determined by the formula

$$\kappa = \frac{c^3}{8\pi^{3/2}\nu_\alpha{}^3 v} \frac{g_n}{g_m},$$

where c is the velocity of light, g_n and g_m are the weights of the stationary states, and v is a function of temperature T:

$$v = \sqrt{\frac{2k_1 T}{m}},$$

where m is the mass of hydrogen and k_1 Boltzmann's constant. The above formula for κ holds when the Doppler effect is the main cause of broadening of the line. For $T = 10{\cdot}000°$ we find $\omega \simeq 10^4$.

It may be mentioned here that there is another cause that will increase the optical thickness of the nebula in the lines of the Balmer series. It is well known that atoms in nebulæ have a tendency to become accumulated in metastable states. The state $2S$ of hydrogen is the metastable one. And the atoms in this state are capable of absorbing the Balmer quanta. If the metastability is a strict one, *i.e.* if the transitions from the metastable state to the lowest states are quite impossible, then Boltzmann's Law

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-\frac{h\nu_2}{kT}}, \tag{40}$$

as Rosseland has shown, will govern the ratio n_2/n_1. But owing to the existence of the small probability of the transition $2S \rightarrow 1S$, the formula for n_2/n_1 will have another form, which we consider in a further paper. It is only important to remark that our values for n_2 are only the lower limits.

In view of these facts, the Balmer series of hydrogen is in a situation quite different from that of other subordinate series. While in all lines of other series the nebula is perfectly transparent (for example in the lines of the Paschen series), in the first members of the Balmer series the optical thickness of the nebula approaches, if not reaches, unity.

Radiation-pressure in the Outer Parts of the Nebula.—The greater part of the ultra-violet radiation of the star is transformed by the nebula into L_α-quanta. The flux of radiation emitted by the nebula will consist chiefly of L_α-quanta. For sufficiently large τ_0 each ultra-violet quantum will give rise to an L_α-quantum, and the flow of L_α-radiation from the nebula will be of the order $\frac{\nu_\alpha}{\nu_c}\pi S$. On the inner surface of the nebula the flux of L_α-radiation will be practically equal to zero, and the flux of ultra-violet radiation will be πS. Now the radiation-pressure in a layer of gas is proportional to the absorption coefficient. On the inner surface of the nebula the flux of radiation consists of ultra-violet quanta, for which the absorption coefficient is small. Therefore the radiation-pressure will not be very large. In the outer parts of the nebula, on the contrary, the flux of radiation consists chiefly of L_α-quanta, and the absorption coefficient is 10^4 times larger than in the case of ultra-violet quanta, while the flux of radiation is of the same order. The radiation-pressure, or, more exactly, the gradient of radiation-pressure, will be here 10^4 times larger than on the inner boundary of the ring. It is physically clear that for τ_0 large the net flux of L_α-radiation πF_α in the outer layer of the ring will be determined by

$$\pi F_\alpha = \left(\frac{r_*}{r_n}\right)^2 \frac{2\pi h\nu_\alpha}{c^2} \int_{\nu_0}^{\infty} \frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}} - 1},$$

where r_* and r_n are respectively the radius of the central star and the

radius of nebula. The average impulse received by a hydrogen atom in normal state per second from L_α-quanta will be

$$\frac{k\pi F_\alpha}{c} = \left(\frac{r_*}{r_n}\right)^2 \frac{2k\pi h\nu_\alpha}{c^3} \int_{\nu_0}^{\infty} \frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}} - 1}.$$

The impulse received by each hydrogen atom from the gravitational field of the central star per second is

$$g(r_*/r_n)^2 m,$$

where g is the gravitational acceleration on the surface of the central star. But not only the normal hydrogen atoms, but the protons also are subject to gravitational force. Therefore the ratio μ of repulsive force R to the attractive force G is given by

$$\mu = \frac{R}{G} = \frac{k\pi}{mg\left(1 + \frac{n^+}{n_1}\right)} \frac{2h\nu_\alpha}{c^3} \int_{\nu_0}^{\infty} \frac{\nu^2 d\nu}{e^{\frac{h\nu}{kT}} - 1},$$

where $\frac{n^+}{n_1}$ is the ratio of the number of protons per cubic centimetre to the number of normal hydrogen atoms in the same volume. Even in the case when we put

$$n^+/n = 500,$$

which value is probably too high,* we obtain for $T = 40{,}000°$

$$\mu = 10^{10}/g.$$

The value of g for the nuclei of planetary nebulæ will be much larger than that for the Sun. But it is very improbable that it may reach 10^{10} cm. sec^{-2}. Therefore we may conclude that when the optical thickness of the nebula in the ultra-violet region is not exceedingly small, the radiation-pressure will be the dominant factor in the exterior parts of the nebula.

Distribution of Light over the Disc.—It would be useless to compute the distribution of ultra-violet light on L_α-radiation over the disc of the nebula, for we have no experimental data on this subject. But it is possible to say something about the distribution of energy over the monochromatic images of the nebula in the higher members of the Balmer series (H_δ, H_ε, $H_{\prime\prime}$, etc.). In fact, the absorption coefficient in these lines will be so small that the nebula will be transparent in these lines in spite of the large number of hydrogen atoms in the second level. If $B'(t')$ is the *Ergiebigkeit* of the nebula in the given Balmer line, and t' is the corresponding optical depth, the distribution of radiation in the higher members of the Balmer series will be governed by the law:

$$I(o, \theta) = C_1 \int_0^{t_0'} B(t') \sec\theta dt',$$

where θ is the angular distance of the point under consideration from the

* Zanstra, *Zs. für Astrophysik*, 2, 337, 1931.

centre of the disc measured from the central star, and C_1 a constant. Owing to the transparency of the nebula, the exponential factor usually occurring in such formulæ will disappear. Integrating we find :

$$I(o, \theta) = C_2 \sec \theta, \tag{41}$$

where C_2 is another constant. Thus the brightness will increase with the distance from the centre of the disc. But for θ large, the brightness will decrease again owing to the curvature of the layers, and a ring of maximum brightness will appear.

It is known from Berman's spectrophotometric investigation * that some planetary nebulæ show rapid decrease of brightness for increasing θ. It seems that the discordance of (41) with these observations could be explained by the assumption that the linear thickness of the shell is of the same order as the radius of the ring. In this case the reduction of the spherical problem to the plane is inadequate, at least for the lines in which the nebula is transparent.

On the other hand, our results concerning the density of L_α-radiation in the central parts of the nebula, and our conclusions about the radiation-pressure, are in some degree independent of the geometrical method adopted.

Poulkovo Observatory :
October 1932.

APPLICATIONS OF DEGENERATE STATISTICS TO STELLAR MATTER.

D. S. Kothari.

(Communicated by Professor R. H. Fowler)

Introduction.—In a recent paper † various formulæ for transport phenomena, *i.e.* thermal and electrical conduction, diffusion and viscosity, have been derived in a form suitable for application to non-degenerate and degenerate (in the sense of Fermi-Dirac) matter. Professor Milne's ‡ investigations on stellar structure have shown the possibility and in certain cases the necessity for the existence of degenerate matter in the stellar interiors. The present paper is divided into five parts. In Part I it is shown that for degenerate matter in stellar interiors the outward flow of energy due to the process of thermal conduction predominates over the flow due to the process of radiation, *i.e.* the energy transfer is mainly conductive and not radiative. The "effective opacity" for degenerate stellar matter is found to be proportional to T^2/ρ^2, where T denotes the temperature and ρ the density, and this should replace the usual $1/T^2$ law

* Berman, *Lick Observatory Bulletin*, **15**, 86, 1930.
† Kothari, *Phil. Mag.*, **13**, 361, 1932. This paper is later referred to as D I.
‡ Milne, *M.N.*, **91**, 1, 1930 ; *M.N.*, **92**, 610, 1932.

which ignores the energy flow by conduction. Some interesting remarks regarding degenerate opacity are noted in this connection. In Part II these results are used to evaluate various physical quantities from formulæ given by Milne in his recent researches. The case of a " model white dwarf," $L = 10^{31}$ ergs/sec., $M = 10^{33}$ grm., is considered at length and the central temperature is found to be of the order of 10^7 degrees, which is in agreement with Milne's * recent ideas.

In Part III it is shown that for degeneracy, whether non-relativistic or relativistic, the gas pressure predominates over the radiation pressure. Gaseous and radiative viscosity is considered in Part IV. The gas viscosity of a degenerate gas is found to be very large compared with what would be computed for a classical gas at the same density and temperature. For degeneracy the radiative viscosity is negligible. In Part V we evaluate, by the usual application of the electromagnetic theory of light-absorption, the opacity coefficient for non-degeneracy and degeneracy from the results previously obtained † for the electrical conductivity. The opacity values so calculated agree in order of magnitude with those obtained in the ordinary way from the method of Kramers and Gaunt.

A comparison between the physical properties of degenerate stellar matter and degenerate terrestrial material (*i.e.* metals) is given in the Appendix.

It is a pleasure to express my grateful thanks to Professor R. H. Fowler and Professor E. A. Milne for many valuable suggestions and constant encouragement during this work. I am also grateful to Dr. R. C. Majumdar for much friendly discussion. To the Government of the United Provinces, India, I am indebted for the award and extension of my research Scholarship.

Part I.—Energy Transfer by Conduction in Stellar Matter

§ 1. The fundamental equation ‡ for the transfer of energy by radiation is

$$H_R = -\frac{c}{\kappa\rho}\frac{dp_R}{dx}, \tag{1}$$

where H_R denotes the net flow by radiation along the x-direction, κ is the opacity coefficient, ρ the density of matter and p_R the radiation pressure.

Substituting

$$p_R = \tfrac{1}{3}aT^4,$$

we get

$$H_R = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dx}$$

$$= -\lambda_R\frac{dT}{dx}, \tag{2}$$

where

$$\lambda_R = \frac{4acT^3}{3\kappa\rho}. \tag{3}$$

* Milne, *Nature*, **128**, 999, 1931. † Kothari, *loc. cit.*

‡ Eddington, *Internal Constitution of Stars* (1926), p. 101. As usual, this is referred to simply as *I.C.S.*

The equation for the flow of energy by thermal conduction is

$$H_c = -\lambda \frac{dT}{dx}, \tag{4}$$

where λ is the thermal conductivity.

The similarity in the forms of (2) and (4) suggests that λ_R may be regarded as " radiational conductivity." *

For a comparison between H_R and H_c we require a knowledge of the thermal conductivity and the opacity coefficient. The values for the conductivity have been already obtained [D I, Table II] and are quoted below in Table I. The opacity coefficient for non-degeneracy was first

TABLE I

	Non-degeneracy	
(i) Opacity.†	$\frac{8\pi^2\theta_2}{315\sqrt{3}\theta_1} \frac{e^6h^2}{c(2\pi m)^{3/2}m_H{}^2R^{7/2}} \frac{Z^2}{A'\mu} \frac{\rho}{T^{7/2}}$,	$4{\cdot}23\ 10^{23}\frac{\rho}{T^{7/2}}$.
(ii) Radiational conductivity, λ_R.	$\frac{105\sqrt{3}\theta_1}{2\pi^2\theta_2} \frac{c^2(2\pi m)^{3/2}am_H{}^2R^{7/2}}{e^6h^2} \frac{A'\mu}{Z^2} \frac{T^{13/2}}{\rho^2}$,	$7{\cdot}22\ 10^{-28}\frac{T^{13/2}}{\rho^2}$.
(iii) Thermal conductivity, λ.	$\frac{2^{13/2}}{\pi^{1/2}} \frac{R^{7/2}}{e^4m^{1/2}Z} \frac{T^{5/2}}{I_1}$,	$3{\cdot}81\ 10^{-5}\frac{T^{5/2}}{I_1}$.
	Degeneracy	
(iv) Opacity.	$\frac{56}{15\sqrt{3}} \frac{e^6}{chR^2} \frac{Z^2}{m_HA'} \frac{1}{T^2}$,	$5{\cdot}02\ 10^{16}\frac{1}{T^2}$.
(v) Radiational opacity, λ_R.	$\frac{5\sqrt{3}}{14} \frac{c^2haR^2m_H}{e^6} \frac{A'}{Z^2} \frac{T^5}{\rho}$,	$6{\cdot}09\ 10^{-21}\frac{T^5}{\rho}$.
(vi) Thermal conductivity, λ.	$\frac{\pi}{8} \frac{h^3R^2}{e^4m^2m_H} \frac{T\rho}{\mu ZI_1}$,	$5{\cdot}43\ 10^{2}\frac{T\rho}{I_1}$.

$\theta_1 = 1{\cdot}0128$. R is the Boltzman's constant (in paper D I we used k).
$\theta_2 = 1{\cdot}0823$. a is the Stefan's constant.
A' is the atomic weight.

* See also Jeans, *Astronomy and Cosmogony* (1928), p. 79. It is interesting to note here, that if η_R denotes the radiative viscosity $\left(\eta_R = \frac{aT^4}{3\kappa\rho c}\right)$ and C_R' the constant volume specific heat for radiation *per unit mass of radiation* $\left(C_R' = \frac{4c^2}{T}\right)$, then

$$\lambda_R / C_R' \eta_R = 1,$$

a relation for radiation corresponding to the well-known relation of similar form for a gas [D I, equation (62)].

† The value given above denotes only the contribution due to the free-free transitions, *i.e.* in Eddington's notation and terminology refers to the case when the " guillotine " falls at a frequency $\nu_1 = E/h$, E being the kinetic energy of the encountering electron (*I.C.S.*, p. 233). Though for the non-degenerate case the contribution due to the bound-free transitions may often be predominant, for simplicity we do not consider it here as it in no way affects essentially the results obtained later.

evaluated by Eddington. The degenerate case has recently been treated by several authors,* who have calculated the contribution to opacity made by free-free electron transitions by following Eddington's treatment with an additional factor to take account of the " already-occupiedness " of the phase cells, and it has been shown † that for sufficiently intense degeneracy the bound-free transitions make a negligible contribution. But it is, however, possible that there might be other factors which (though making an inappreciable contribution in the non-degenerate case) in this case of degeneracy make an appreciable contribution, and hence the value of the opacity quoted in the following table is to be regarded as denoting only its lower limit.‡ Whenever numerical values are given below, they refer to the case of fully ionised iron-atoms ($Z=26$) and the mean molecular weight $\mu=2{\cdot}1$.

As shown in [D I, equation (68)], I_1 stands for

$$I_1=2\pi\log_e(1+\alpha_0{}^2), \tag{5}$$

where $\alpha_0=8{\cdot}45\ 10^{-7}\dfrac{T}{\rho^{1/3}}$ for non-degeneracy and $\alpha_0=5{\cdot}55\ 10^{-2}\rho^{1/3}$ for degeneracy. The values of I_1 and α_0 for different values of ρ are given § in the paper D I.

The following relations, which are obvious from the values given in the table, may be noted:—

$$\frac{\text{Opacity (non-deg.)}}{\text{Opacity (deg.)}}=\frac{2\pi^2}{21}\frac{\theta_2}{\theta_1}\frac{A_0}{7}=0{\cdot}143A_0, \tag{6}$$

$$\frac{\text{Radiational conductivity (non-deg.)}}{\text{Radiational conductivity (deg.)}}=\frac{21\theta_1}{2\pi^2\theta_2}\frac{7}{A_0}=\frac{6{\cdot}97}{A_0}, \tag{7}$$

$$\frac{\text{Thermal conductivity (non-deg.)}}{\text{Thermal conductivity (deg.)}}=\frac{128}{\pi^3}\frac{1}{A_0}=\frac{4{\cdot}13}{A_0}, \tag{8}$$

where A_0 is the Sommerfeld degeneracy discriminant.‖

$$A_0=\frac{nh^3}{2(2\pi mRT)^{3/2}}=\frac{h^3}{2(2\pi mRT)^{3/2}}\frac{\rho}{\mu m_H}=5{\cdot}88\ 10^7\frac{\rho}{T^{3/2}}. \tag{9}$$

n is the number of electrons per unit volume.

We may note in connection with these formulæ given in the table, that

(i) they, being based on the usual approximations which are made to avoid mathematical complications, are not valid for " incipient "

* Majumdar, *A.N.*, **243**, 6, 1931 ; Chandrasikhar, *Proc. Roy. Soc.*, A, **133**, 241, 1931 ; Swirles, *M.N.*, **91**, 861, 1931 ; Kothari and Majumdar, *A.N.*, **244**, 66, 1931.

† Kothari and Majumdar, *loc. cit.*

‡ The scattering due to free electrons may perhaps be of importance [" excess scattering " : S. Rosseland, *Astrophysik*, Berlin (1931)].

§ See Table II given later.

‖ To avoid ambiguity, we use A for the Fermi-Dirac-distribution constant and A_0 for the degeneracy discriminant. A for non-degeneracy and degeneracy approaches asymptotically A_0 and $\text{Exp}\left\{\left(\frac{3\pi^{1/2}}{4}\right)^{2/3}A_0{}^{2/3}\right\}$ respectively.

degeneracy ($A_0 \sim 1$) but are applicable only when the degeneracy or non-degeneracy is sufficiently intense (*i.e.* $A_0 >> 1$ or $A_0 << 1$);

(ii) we have used throughout the non-relativistic statistics, and hence these formulæ are applicable so long as the relativistic effect is negligible, and this is so provided *

$$\frac{h^2}{mc}\left(\frac{3n}{8\pi}\right)^{1/3} << 1 \text{ in the case of degeneracy,}$$

and

$$\frac{mc^2}{RT} >> 1 \text{ in the case of non-degeneracy,}$$

or, in terms of A_0,

$$T << \frac{mc^2}{2R}\left(\frac{3\pi^{1/2}}{4}\right)^{2/3}\frac{1}{{A_0}^{2/3}}$$

$$<< \frac{3{\cdot}57\ 10^9}{{A_0}^{2/3}}, \text{ for degeneracy,}$$

and

$$T << 5{\cdot}90\ 10^9, \text{ for non-degeneracy.}$$

The degenerate opacity is independent of the density and so is also the thermal conductivity in the non-degenerate case. The essential points are illustrated in fig. 1, where the values for non-degenerate and degenerate transparency (reciprocal of opacity) and conductivity for a given temperature are represented against density (the scale used is logarithmic). The straight line AB is the locus of the points where the straight lines representing the non-degenerate and degenerate transparency cut each other. The equation to AB is

$$\log 1/\kappa = -0{\cdot}80 + 4/3 \log 10^{-5}\rho.$$

Along AB we can mark a " ladder " of points each labelled with the corresponding value of the temperature. By selecting on AB the point corresponding to any particular temperature T and drawing through this point straight lines parallel to HE and HF, we get a representation of transparency against density for the temperature T. It may be noted that in the neighbourhood of H the degeneracy is " incipient " and so near H the lines HE and HF do not give the transparency accurately. In fact the straight lines HE and HF are the asymptotes of the actual transparency-density curve.

The straight line A′B′ is the locus of the points where the lines representing the non-degenerate and degenerate conductivity cut each other. The equation for A′B′ is

$$\log \lambda = 15{\cdot}81 + \tfrac{5}{3} \log 10^{-5}\rho.$$

A " ladder " of temperature values is marked along A′B′.

* Kothari, *Phil. Mag.*, 12, 675, 1931.

From an inspection of the figure, it is at once seen that:

(i) *A degenerate gas is more opaque (less transparent) than a " classical gas " (non-degenerate) at the same temperature.*

A degenerate gas is more conducting than a " classical gas " at the same temperature.

(ii) *The opacity of a degenerate gas is very small compared with what would be computed for a " classical gas " at the same temperature and density.*

The conductivity of a degenerate gas is very large compared with what would be computed for a " classical gas " at the same temperature.

(iii) *The opacity of a " classical gas " is very small compared with what would be computed for a degenerate gas at the same temperature.*

The conductivity of a " classical gas " is very large compared with what would be computed for a degenerate gas at the same density and temperature.

It may particularly be noted that a degenerate gas at any given temperature T has smaller transparency than the " classical gas " at any density and temperature not below T. Further, we can put together the statements (ii) and (iii) as: the actual transparency and conductivity of a gas in one phase is larger than what would be computed for the same temperature and density by using the formulæ valid for the other phase.*

§ 2. We now proceed to the comparison between H_R and H_c.

$$\frac{H_R}{H_c}=\frac{\lambda_R}{\lambda}. \tag{10}$$

---

* The statement often made that a degenerate gas is more transparent than the " classical gas " is very misleading and has no significance.

At first sight statement (iii) may appear paradoxical, but on examination it at once becomes clear. We may illustrate the point by a familiar example—the mean kinetic energy per electron. We have

$$\frac{\text{Mean K.E. for degenerate gas}}{\text{Mean K.E. for " classical gas "}}=\frac{3}{10}\left(\frac{3n}{8\pi}\right)^{2/3}\frac{h^2}{m}\Big/\frac{3}{2}RT=\left(\frac{9\pi}{5^3\times 2}\right)^{1/3}A_0^{2/3},$$

and the corresponding three statements for this case are:

(i) The mean K.E. per electron is greater for degenerate gas than for " classical gas " at the same temperature.

(ii) The mean K.E. per electron for a degenerate gas is very large compared with what would be computed for a " classical gas " at the same temperature.

(iii) The mean K.E. per electron for " classical gas " is very large compared with what would be computed for a degenerate gas at the same density.

So far as I know, the statement (ii)—for the case of opacity—has always been emphasised, while (i) and (iii) have not been (at least explicitly) noted before. They are really obvious, but still they serve to illustrate some interesting points. For example, the fact that in metals the electron gas is degenerate and they are more opaque and better conducting than non-metals is a good illustration of the statement (i) above.

The reciprocity of (ii) and (iii) may be noted. (In fact any one of them is implied in any other.) See a note on this to appear in *Phil. Mag.*

We treat first the non-degenerate case.

Case (a). Non-degeneracy. $A_0 << 1$.—Putting the values for λ_R and λ from Table I (ii) and (iii),

$$\frac{H_R}{H_c} = \frac{105\sqrt{3}\theta_1}{64\theta_2} \frac{ac^2m^2m_H{}^2}{e^2h^2} \frac{A'\mu}{Z} \frac{T^4}{\rho^2} I_1 \tag{11}$$

$$= 1{\cdot}90\ 10^{-23} \frac{T^4}{\rho^2} I_1. \tag{12}$$

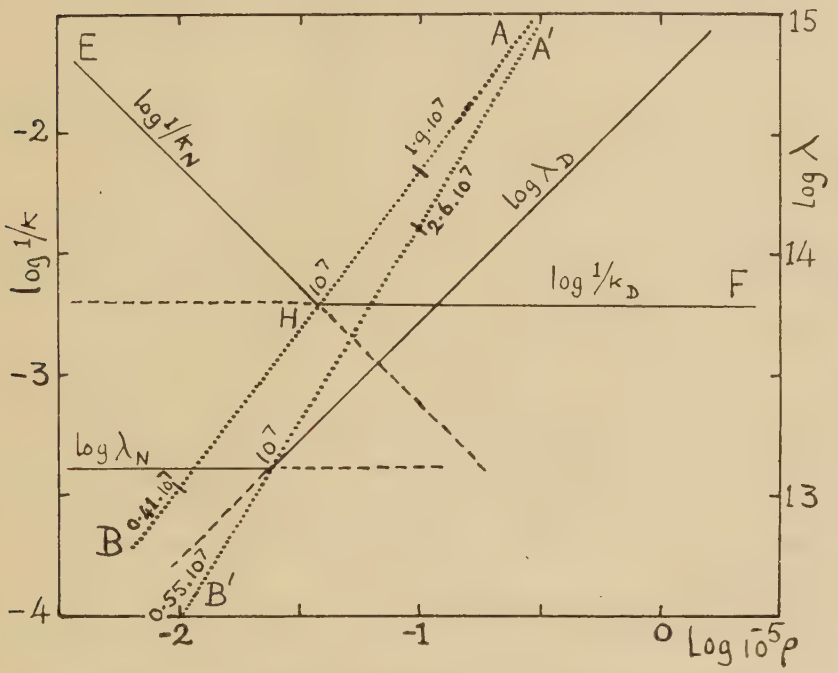

FIG. 1.—*Suffixes* N *and* D *signify Non-degenerate and Degenerate cases respectively.*

The lines EH *and* HF *representing Transparency and similarly the lines representing Conductivity that are drawn in the figure are for the temperature of* 10^7 *degrees and for fully ionised iron atoms ;* $Z = 26$, $\mu = 2{\cdot}1$.

Expressing (11) in terms of A_0 we have,

$$\frac{H_R}{H_c} = \frac{105\sqrt{3}\theta_1}{64 \times 32\pi^3\theta_2} \frac{ac^2h^4}{e^2mR^3} \frac{A'}{\mu Z} \frac{T}{A_0{}^2} I_1 \tag{13}$$

$$= 6{\cdot}55\ 10^{-8} \frac{T}{A_0{}^2} I_1. \tag{14}$$

As an illustration let us apply this result to the centre of a star built on Eddington's perfect gas model. For the brighter component of Capella, Eddington * has obtained for T_c (central temperature) and ρ_c (central densities) the values

$$T_c = 9{\cdot}08\ 10^6 \text{ degrees},$$
$$\rho_c = 0{\cdot}1234 \text{ grm./c.c.}$$

* Eddington, *I.C.S.*, p. 145.

Using (5) and (9) we find A_c (value of A_0 at the centre) and I_1, and thus H_R/H_c for the centre of the star.

Thus for Capella (brighter component)

$$A_c = 2{\cdot}6\ 10^{-4},$$
$$I_1 = 34{\cdot}4,$$
$$H_R/H_c = 2{\cdot}9\ 10^{8}.$$

We tabulate below the values of H_R/H_c for the centres of a few stars worked out by Eddington on his theory (*I.C.S.*, chap. vii).

Name	T_c	ρ_c	A_c	H_R/H_c (at the Centre)
	$\times 10^6$			
δ Cephei	6·61	0·0185	$6{\cdot}4\ 10^{-5}$	$4{\cdot}1\ 10^{9}$
Capella	9·08	0·1234	$2{\cdot}6\ 10^{-4}$	$2{\cdot}9\ 10^{8}$
V Puppis... ...	42·4	3·35	$7{\cdot}1\ 10^{-4}$	$2{\cdot}2\ 10^{8}$
Sun	39·5	76·5	$1{\cdot}8\ 10^{-2}$	$2{\cdot}0\ 10^{5}$
Kreuger 60 ...	32·2	493	$1{\cdot}6\ 10^{-1}$	$1{\cdot}4\ 10^{3}$

Thus, on Eddington's model the energy transfer is practically wholly radiative and this remains so even when H_R, the energy flow due to radiation, is reduced by about a factor of ten so as to allow for the fact that the "astronomical opacity" required in Eddington's theory is about ten times the "theoretical opacity." * We note that H_R/H_c decreases as the central density increases.

We now treat the degenerate case.

Case (*b*). *Degeneracy.* $A_0 >> 1$.—From (10) and Table I (v) and (vi), we have

$$\frac{H_R}{H_c} = \frac{5\sqrt{3}\pi}{112}\,\frac{ac^2m^2m_H{}^2}{e^2h^2}\,\frac{A'\mu}{Z}\,\frac{T^4}{\rho^2}I_1 \tag{15}$$

$$= 1{\cdot}12\ 10^{-23}\frac{T^4}{\rho^2}I_1. \tag{16}$$

Expressing in terms of A_0,

$$\frac{H_R}{H_c} = \frac{5\sqrt{3}}{112\times 32\pi^2}\,\frac{ac^2h^4}{e^2mR^3}\,\frac{A'}{\mu Z}\,\frac{T}{A_0{}^2}I_1 \tag{17}$$

$$= 3{\cdot}88\ 10^{-8}\frac{T}{A_0{}^2}I_1, \tag{18}$$

where I_1 is given by (5). The values of I_1 and α_0 for different values of the density are given on p. 69. (This is taken from D I.)

As $A_0 >> 1$ for degeneracy, it is immediately obvious from (18) that the energy transfer by conduction is comparable to that by radiation, and as

* Calculated on the assumption that there is no "hydrogen abundance."

the degeneracy gets intense (*i.e.* A_0 becomes larger) the conductive flow increases over the radiative flow.

TABLE II

(*Degenerate Case*)

ρ grm./c.c.	a_0	I_1	I_2
10^6	5·55	21·7	21·9
10^5	2·58	12·8	13·6
10^4	1·20	5·6	8·2
10^3	0·56	1·7	6·5
10^2	0·26	0·43	6·3
10	0·12	0·09	6·3
1·0	0·06	0·02	6·3

From (13) and (17) we note that

$$\frac{(H_R/H_c)\text{ non-deg.}}{(H_R/H_c)\text{ deg.}} = \frac{147\pi}{256}\frac{\theta_1}{\theta_2} = 1{\cdot}69. \tag{19}$$

§ 3. *Effective Opacity.**—Let H denote the total net flow of energy, then

$$H = H_R + H_c \tag{20}$$

$$= -\left(\frac{c}{\kappa\rho}\frac{dp_R}{dx} + \lambda\frac{dT}{dx}\right) \tag{21}$$

$$= -\frac{4acT^3}{3\kappa\rho}\left\{1 + \frac{\lambda}{\frac{4acT^3}{3\kappa\rho}}\right\}\frac{dT}{dx}.$$

$$H = -\frac{4acT^3}{3\kappa'\rho}\frac{dT}{dx} = -\frac{c}{\kappa'\rho}\frac{dp_R}{dx}, \tag{22}$$

where κ' may be regarded as the " effective opacity."

$$\kappa' = \frac{\kappa}{1 + \lambda/\lambda_R}. \tag{23}$$

When the radiative flow is small compared to the conductive flow, *i.e.*

$$\frac{H_R}{H_c} = \frac{\lambda_R}{\lambda} << 1, \tag{24}$$

then we have as an approximation,

$$\kappa' \doteqdot \frac{\kappa\lambda_R}{\lambda} = \frac{4}{3}\frac{acT^3}{\rho\lambda}, \tag{25}$$

which on putting the value of λ becomes

$$\kappa' \doteqdot \frac{32}{3\pi}\frac{ace^4m^2m_H}{h^3R^2}\mu Z\frac{T^2}{\rho^2}I_1. \tag{26}$$

$$\boxed{\kappa' \doteqdot 5{\cdot}63\ 10^{-7}\left(\frac{\mu}{2{\cdot}1}\right)\frac{T^2}{\rho^2}I_1}$$

* The idea of using an effective opacity was kindly suggested to me by Milne.

Thus the effect of conduction is simply taken account of by using the "effective opacity" κ' instead of the actual opacity κ in the fundamental equation of radiative flow, and κ' is found to be $=\text{constant}\times\left(\frac{T}{\rho}\right)^2$ (ignoring the variation of I_1 with ρ).

Part II.—Numerical Calculations for "model White Dwarf"

We shall now apply the results of Part I to obtain some "insight" regarding the conditions in the interior of the white dwarfs.* In § 1 and 2 various formulæ for calculating the physical quantities are given † and their values for certain limiting cases are discussed. In § 3 we apply these results to the case of a "model star" and evaluate the physical quantities for various sizes of the degenerate core. In § 4 we make a "reasonable" estimate of the size of the core and thus find "reasonable" values of the physical quantities for the "model white dwarf." In § 5 we discuss as to how far the values so calculated afford insight regarding the conditions existing in actual white dwarfs.

The white dwarfs have a smaller luminosity than other stars of the same mass and spectral class. The following table (reproduced from the Halley Lecture,* p. 10) gives particulars of four white dwarfs of which we have knowledge.

TABLE III

The Known White Dwarfs

Name	Type	Vis. Mag.	Parallax	Abs. Vis. Mag.	Effective Temperature, T_e	Abs. Bol. Mag.	Log L. L = Luminosity in erg/sec.	Radius. Unit, Sun's Radius	Mass. Unit, Sun's Mass	Mean Density, ρ_m	L/M ergs/grm. sec.
Sirius B	A7	8·4	·377	11·3	8,000	11·2	31·04	·027	·85	5×10^4	·007
α_2 Eridani B	Ao	9·7	·203	11·2	11,200	10·8	31·20	·018	·44	$\cdot98\times10^5$	·02
Procyon B	? F	13·5	·312	16·0	? 7,000	[15]	[29·52]	? ·009	·37	7×10^5 ?	[·0005]
Van Maanen's star	F	12·3	·255	14·3	7,000	14·3	29·80	·009	(? ·25)	5×10^5 ?	(? ·001)

Numbers in brackets [ ] are uncertain. Numbers marked with ? are conjectural.
Numbers in brackets ( ) marked with ? are derived from conjectural quantities.
The companion of *o* Ceti and Oosterhoff's star may be added to the above list.
Log L is calculated from the usual relation, $\log L = -0{\cdot}4M_{\text{Bol.}} + 35{\cdot}52$. (The value of L/M for α_2 Eridani B given in the Halley Lecture as ·002 is a misprint.)
For comparison we may note :

Sun's mass $=2{\cdot}0\times10^{33}$ (grm.). Abs. Bol. mag. $=4{\cdot}85$.
Radius $=6{\cdot}95\times10^{10}$ (cm.). Luminosity $=3{\cdot}78\times10^{33}$ (ergs/sec.).
Mean density $=1{\cdot}41$ (grm./cm.^{3}). Effective temperature $=5741$ degrees.

* An up-to-date and lucid account of the white dwarf stars is given in the Halley Lecture (1932) by Milne.

† These are mostly taken from Milne, *M.N.*, **92**, 610, 1932.

In the following calculation we shall take a " model white dwarf " of mass $M = 10^{33}$, luminosity $L = 10^{31}$, and therefore $L/M = 10^{-2}$ ergs/grm. sec. The central temperature and other physical quantities are calculated from the formulæ given by Milne * in his recent detailed study of the *two-phase configurations of the " generalised standard model."* " The ' generalised standard model ' is defined as a steady-state aggregate of given mass M and luminosity L, in which the source strength is uniform and equal to L/M per gram throughout the system and in which the opacity is constant in each phase, though not necessarily the same constant in different phases " [M II, p. 611].

Because of these restrictions, *i.e.*

(i) the star is treated as built on the "generalised standard model," and

(ii) the number of possible phases is confined to only two: non-relativistic classical gas envelope and non-relativistic degenerate core,

we cannot regard the results obtained for the central temperature, etc., as representing the actual conditions in the observed white dwarfs; but these results do afford some " insight " as regards the actual conditions.†

§ 1. The formulæ derived for the generalised standard model which are required here are taken from Milne's paper (M II) and we use his notation except for the slight alterations ‡ that for the mean molecular weight we use here μm_H and not μ, and the degeneracy constant $K/\mu^{5/3}$ used here is Milne's K.

The relations connecting temperature and density in the two phases are

$$\frac{R\rho}{\mu m_H} = \tfrac{1}{3}aT^3 \frac{\beta_1}{1-\beta_1} \text{ for the perfect gas envelope,} \tag{27}$$

$$\frac{K\rho^{5/3}}{\mu^{5/3}} = \tfrac{1}{3}aT^4 \frac{\beta_2}{1-\beta_2} + \frac{(R/m_H)^{20/3}}{(\frac{1}{3}a)^{5/3}K^4} \frac{(1-\beta_1)^{5/3}(\beta_1-\beta_2)}{\beta_1{}^{8/3}(1-\beta_2)} \text{ for the degenerate core,} \tag{28}$$

and the temperature and density at the interface § between the envelope and the core are given by

$$T' = \frac{(R/m_H)^{5/3}}{(\frac{1}{3}a)^{2/3}K}\left(\frac{1-\beta_1}{\beta_1}\right)^{2/3}, \tag{29}$$

$$\rho' = \frac{(R/m_H)^4\mu}{\frac{1}{3}aK^3} \frac{1-\beta_1}{\beta_1}, \tag{30}$$

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* *M.N.*, **92**, 610, 1932. This paper will be referred to as M II.

† See M II, pp. 614, 639; also *Zs.f. Astrophysik*, **4**, 86, 1932, for some important remarks in this connection.

‡ This is done to show explicitly the dependence on μ. The pressure of degenerate gas $= \dfrac{K}{\mu^{5/3}}\rho^{5/3}$.

§ At the interface the pressure given by the degenerate and non-degenerate formulæ are equal, $\dfrac{K\rho'^{5/3}}{\mu^{5/3}} = \dfrac{RT'}{\mu m_H}$. This gives, by putting for $K = \dfrac{8\pi h^2}{15m}\left(\dfrac{3}{8\pi m_H}\right)^{5/3}$, the value of A_0 (Sommerfeld's degeneracy discriminant).

$$A_0 = \left(\frac{250}{27\pi}\right)^{1/2} = 2{\cdot}97 \text{ at the interface.}$$

where

$$\beta_1 = 1 - \frac{\kappa_1 L}{4\pi cGM},$$

$$\beta_2 = 1 - \frac{\kappa_2 L}{4\pi cGM},$$

κ_1 and κ_2 being the opacities in the gaseous and the degenerate phase respectively.

The solution of the Emden equation (index 3)

$$\frac{1}{\xi^2}\frac{d}{d\xi^2}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^3 = 0, \tag{31}$$

to be selected in the envelope depends solely on the value of the discriminant C.

$$C = \frac{16(R/m_H)^4}{\pi^{\frac{1}{3}} aG^3M^2\mu^4}\,\frac{1-\beta_1}{\beta_1^4} \tag{32}$$

(C depends only on the " observables " L, M and κ_1), and the solution of the Emden equation (index 3/2)

$$\frac{1}{\eta^2}\frac{d}{d\eta^2}\left(\eta^2\frac{d\psi}{d\eta}\right) + \psi^{3/2} = 0, \tag{33}$$

for the core is the "Emden-solution" because the degenerate phase is assumed to extend (in the physical sense) up to the centre of the model. The solution in the envelope is of the "collapsed," "Emden" or "centrally-condensed" type according as

$$C^{-1/2} > = \text{ or } < \omega_3{}^0, \qquad \omega_3{}^0 = 2{\cdot}0181 \ldots \tag{34}$$

If $f(\xi)$ be the solution of the Emden equation for the envelope, $f(\xi)$ vanishing at $\xi = \xi_0$, and if $\phi(\eta)$ be the Emden solution for the core, $\phi(\eta)$ vanishing at $\eta = \eta_0$, then the physical quantities (for example, the radius of the configuration, mean density, central density, etc.) can all be expressed in terms of invariant form, a/ξ_0, b/η_0, $af(a)$, $b^4\phi(b)$, $-a^2f'(a)$, $-b^5\phi'(b)$, so as to be independent of the " normalisation " to the zeros ξ_0, η_0 ; where the values of a and b are obtained by solving the two equations of fit

$$\frac{a[f(a)]^3}{f'(a)} = \frac{b[\phi(b)]^{3/2}}{\phi'(b)}, \tag{35}$$

$$\frac{af'(a)}{f(a)} = \frac{5}{8}\frac{\beta_1}{\beta_2}\frac{b\phi'(b)}{\phi(b)}. \tag{36}$$

The equations of fit have an explicit approximate solution when $C^{2/3}\left(\frac{5}{8}\frac{\beta_1}{\beta_2}\right)$ is small ; we then have

$$a = \xi_0\left[1 - C^{2/3}\left(\frac{5}{8}\frac{\beta_1}{\beta_2}\right)\omega_{3/2}^{1/3} + \ldots\right], \tag{37}$$

$$b = \eta_0\left[1 - C^{2/3}\left(\frac{5}{8}\frac{\beta_1}{\beta_2}\right)^2\omega_{3/2}^{1/3} + \ldots\right]. \tag{38}$$

In other cases the solution is obtained by graphical methods and for a few values of C the solutions are given in M II.

The formulæ for the physical quantities we require are tabulated below. Owing to the existence of the equations of fit, transformations of these formulæ are possible. The form given in Column I is suited for the case when $a \sim 0$ (core very small) and that given in Column II for the case when $a \sim \xi_0$ (envelope very thin).

TABLE IV

(1) $$\frac{\text{Radius of core}}{\text{Radius of configuration}} = \frac{a}{\xi_0}.$$

(2) Radius of configuration r_1.	
$r_1 = \dfrac{(\frac{1}{3}a)^{1/6}}{(\pi G)^{1/2}} \dfrac{K}{(R/m_H)^{2/3}\mu} \dfrac{\xi_0 f(a)}{\beta_1^{1/3}(1-\beta_1)^{1/6}}$	$= \dfrac{5K/\mu^{5/3}}{2^{7/3}\pi^{2/3}GM^{1/3}\beta_2} \dfrac{\xi_0}{a}\left[-\dfrac{\xi_0^2 f'(\xi_0)}{a^2 f'(a)} b^5\varphi'(b)\right]^{1/3}.$
(3) Central density ρ_c.	
$\rho_c = \dfrac{(R/m_H)^4\mu}{\frac{1}{3}aK^3} \dfrac{1-\beta_1}{\beta_1} \dfrac{[\sigma_{3/2}]^{3/2}}{[\eta_0^4\varphi(b)]^{3/2}}$	$= \dfrac{32}{125} \dfrac{\pi G^3 M^2 \mu^5 \beta_2^3}{K^3} [\sigma_{3/2}]^{3/2}\left[-\dfrac{a^2 f'(a)(b/\eta_0)^3}{\xi_0^2 f'(\xi_0) b^5\varphi'(b)}\right]^2.$
(4) $\dfrac{\text{Mean density } \rho_m}{\text{Central density } \rho_c}$.	
$\dfrac{\rho_m}{\rho_c} = -3\dfrac{[\eta_0^4\varphi(b)]^{3/2}}{[\xi_0 f(a)]^3} \dfrac{\xi_0^2 f'(\xi_0)}{[\sigma_{3/2}]^{3/2}}$	$= -3\left(\dfrac{\eta_0}{b}\right)^6 \dfrac{b^5\varphi'(b)}{[\sigma_{3/2}]^{3/2}} \left(\dfrac{a}{\xi_0}\right)^3 \dfrac{\xi_0^2 f'(\xi_0)}{a^2 f'(a)}.$

(5) Central temperature T_c.

$$T_c = \frac{(R/m_H)^{5/3}}{(\frac{1}{3}a)^{2/3}K}\left(\frac{1-\beta_1}{\beta_1}\right)^{5/12}\left(\frac{1-\beta_2}{\beta_2}\right)^{1/4}\left[\left(\frac{\sigma_{3/2}}{\eta_0^4\varphi(b)}\right)^{5/2} - \frac{\beta_1-\beta_2}{\beta_1(1-\beta_2)}\right]^{1/4}$$

$$= \left[\frac{1-\beta_2}{\beta_2}\frac{K}{\frac{1}{3}a}\left(\frac{32\pi}{125}\frac{G^3M^2}{K^3}[\sigma_{3/2}]^{3/2}\beta_2^3\mu^4\right)^{5/3}\left(-\frac{a^2f'(a)(b/\eta_0)^3}{\xi_0^2f'(\xi_0)b^5\varphi'(b)}\right)^{10/3} - \frac{(R/m_H)^{20/3}}{(\frac{1}{3}a)^{8/3}K^4}\frac{(1-\beta_1)^{5/3}(\beta_1-\beta_2)}{\beta_1^{8/3}\beta_2}\right]^{1/4}.$$

$$\sigma_{3/2} = \eta_0^4\varphi(0\,;\ \omega_{3/2}^0) = 178{\cdot}23.$$
$$\omega_{3/2}^0 = -\eta_0^5\varphi'(0\,;\ \omega_{3/2}^0) = 132{\cdot}39.$$
$$\sigma_3 = \xi_0 f(0\,;\ \omega_3^0) = 6{\cdot}8969.$$
$$\omega_3^0 = -\xi_0^2 f'(0\,;\ \omega_3^0) = 2{\cdot}0181.$$

Milne has further shown from a study of the solutions of the equations of fit that there exists a critical mass $\mathfrak{M}$, where

$$\omega_{3/2}^0 = 2{\cdot}0181\ \ldots\ = \left[\frac{16(R/m_H)^4}{\pi\frac{1}{3}aG^3\mu^4\mathfrak{M}^2}\frac{1-\beta_1}{\beta_1^4}\right]^{-1/2}_{\beta_1=4/5},$$

which gives

$$\frac{\mathfrak{M}}{\odot} = 2{\cdot}88\left(\frac{2{\cdot}1}{\mu}\right)^2. \tag{39}$$

(i) For $M \leqslant \mathfrak{M}$, all configurations which exist are of collapsed type, and β_1 necessarily lies between β_1^0 and unity, where $4/5 < \beta_1^0 < 1$; β_1^0 being a function of M given by

$$\omega_3^0 = 2{\cdot}0181\ \ldots\ = \left[\frac{16}{\pi}\frac{(R/m_H)^4}{\frac{1}{3}aG^3\mu^4M^2}\frac{1-\beta_1^0}{(\beta_1^0)^4}\right]^{-1/2}, \tag{40}$$

i.e. by the quartic equation

$$1-\beta_1{}^0 = 6{\cdot}00\ 10^{-2}\left(\frac{M}{\odot}\right)^2\left(\frac{\mu}{2{\cdot}1}\right)^4(\beta_1{}^0)^4. \qquad (41)^*$$

(ii) For $M > \mathfrak{M}$, the configurations which exist are either collapsed or centrally condensed. In this paper we consider only the case of $M < \mathfrak{M}$.

For the two-phase collapsed configurations the radius for a given mass is uniquely fixed by the value of $(1-\beta_1)$. As $(1-\beta_1)$ varies from zero to $(1-\beta_1{}^0)$, the configurations pass from one which is all core to one in which the central density is just large enough for degeneracy to be setting in. This linear series of collapsed configurations passes continuously through the above limiting configuration which is just degenerate at the centre, to the diffuse Emden-Eddington polytropes † of arbitrary radius.

It is of interest to enumerate and compare here the values for the physical quantities in the two limiting cases of the two-phase configuration ; (i) when it is all core $(1-\beta_1=0)$ and (ii) when the core just vanishes $(\beta_1=\beta_1{}^0)$. The results are easily obtained from Table IV and using freely (40).‡

(*a*) Radius of configuration (r).

$$r\ (\text{No core})\S = \frac{\sigma_3(\omega_3{}^0)^{1/3}}{\pi^{1/2}}\left(\frac{16}{\pi}\right)^{1/6}\frac{K}{G\mu^{5/3}}\frac{1}{\beta_1{}^0 M^{1/3}} = 2{\cdot}0\ 10^9\frac{1}{\beta_1{}^0}\left(\frac{\odot}{M}\right)^{1/3}. \qquad (42)$$

$$r\ (\text{All core}) = \frac{5(\omega_{3/2}{}^0)^{1/3}}{2^{7/3}\pi^{2/3}}\frac{K}{G\mu^{5/3}}\frac{1}{\beta_2 M^{1/3}} = 8{\cdot}1\ 10^8\frac{1}{\beta_2}\left(\frac{\odot}{M}\right)^{1/3}. \qquad (43)$$

$$\frac{r\ (\text{No core})}{r\ (\text{All core})} = \left(\frac{\omega_3{}^0}{\omega_{3/2}{}^0}\right)^{1/3}\frac{8\beta_2}{5\beta_1{}^0}\sigma_3 = 2{\cdot}73\frac{\beta_2}{\beta_1{}^0}. \qquad (44)$$

(*b*) Central density (ρ_c).

$$\rho_c\,(\text{No core}) = \frac{\pi}{16(\omega_3{}^0)^2}\frac{G^3\mu^5}{K^3}(\beta_1{}^0)^3M^2 = 2{\cdot}4\ 10^6(\beta_1{}^0)^3\left(\frac{M}{\odot}\right)^2. \qquad (45)$$

$$\rho_c\,(\text{All core}) = \frac{32\pi}{125}\frac{(\sigma_{3/2})^{3/2}}{(\omega_{3/2}{}^0)^2}\frac{G^3\mu^5}{K^3}\beta_2{}^3M^2 = 5{\cdot}4\ 10^6\beta_2{}^3\left(\frac{M}{\odot}\right)^2. \qquad (46)$$

$$\frac{\rho_c\,(\text{No core})}{\rho_c\,(\text{All core})} = \frac{125}{512}\left(\frac{\omega_{3/2}{}^0}{\omega_3{}^0}\right)^2\frac{1}{(\sigma_{3/2})^{3/2}}\left(\frac{\beta_1{}^0}{\beta_2}\right)^3 = 0{\cdot}442\left(\frac{\beta_1{}^0}{\beta_2}\right)^3. \qquad (47)$$

(*c*) $\dfrac{\text{Mean density } \rho_m}{\text{Central density } \rho_c}$.

$$\frac{\rho_m}{\rho_c}\,(\text{No core}) = \frac{3\omega_3{}^0}{(\sigma_3)^3} = \frac{1}{54{\cdot}2}\ ;\quad \frac{\rho_m}{\rho_c}\,(\text{All core}) = \frac{3\omega_{3/2}{}^0}{(\sigma_{3/2})^{3/2}} = \frac{1}{5{\cdot}99}.$$

$$\frac{(\rho_m/\rho_c)\ \text{No core}}{(\rho_m/\rho_c)\ \text{All core}} = \frac{\omega_3{}^0}{\omega_{3/2}{}^0}\left(\frac{\sigma_{3/2}}{\sigma_3{}^2}\right)^{3/2} = \frac{1}{9{\cdot}05}. \qquad (48)$$

* For the solution of this equation we can utilise *I.C.S.*, Table 14, p. 137.

† M, L and κ_1 for Emden-Eddington polytropes satisfy (40).

‡ The values for any composite two-phase collapsed configuration will lie between these two limits.

§ For brevity we write " no core " to denote the configuration in which the core *just* vanishes.

(*d*) Central temperature (T_c).

(i) $$T_c(\text{No core}) = \left(\frac{\pi}{16(\omega_3{}^0)^2}\right)^{2/3} \frac{G^3\mu^{8/3}}{K(R/m_H)} (\beta_1{}^0)^2 M^{4/3}$$
$$= 1{\cdot}3\ 10^9 (\beta_1{}^0)^2 \left(\frac{M}{\odot}\right)^{4/3}. \qquad (49)$$

(ii) $$T_c(\text{All core}) = \frac{(\sigma_{3/2})^{5/8}}{(\omega_{3/2}{}^0)^{5/6}} \frac{(4\pi)^{1/6}(2/5)^{5/4}}{(\frac{1}{3}ac)^{1/4}K} G\mu^{5/3} M^{7/12} (\kappa_2 L)^{1/4} \beta_2$$
$$= 2{\cdot}9\ 10^8 \left(\frac{M}{\odot}\right)^{5/6} \left(\frac{\kappa_2 L}{M}\right)^{1/4} \beta_2. \qquad (50)$$

$$\frac{T_c(\text{No core})}{T_c(\text{All core})} = 4{\cdot}43 \left(\frac{\mu}{2{\cdot}1}\right) \frac{(\beta_1{}^0)^2}{\beta_2} \left(\frac{M}{\odot}\right)^{1/2} \left(\frac{\kappa_2 L}{M}\right)^{-1/4}. \qquad (51)$$

We shall denote this ratio by γ^{-1}. γ decreases as $\frac{L}{M}$ becomes smaller.

§ 2. *Central Temperature* (T_c).—The formula given for T_c in Table IV is easily put in the form (or directly from (28))

$$T_c{}^4 = \frac{1-\beta_2}{\beta_2} \left\{ \frac{K\rho_c{}^{5/3}}{\frac{1}{3}a\mu^{5/3}} - \mathcal{J}^4 \frac{(1-\beta_1)^{5/3}}{\beta_1{}^{8/3}} \right\} + \frac{\mathcal{J}^4}{\beta_2} \left(\frac{1-\beta_1}{\beta_1}\right)^{8/3}, \qquad (52)$$

$$\mathcal{J} = \frac{(R/m_H)^{5/3}}{(\frac{1}{3}a)^{2/3}K} = 8{\cdot}48\ 10^9. \qquad (53)$$

Putting
$$1-\beta_2 = \frac{\kappa_2 L}{4\pi cGM},$$

$$T_c{}^4 = \frac{L\kappa_2}{M\beta_2} \left\{ 4{\cdot}49\ 10^{22} \left(\frac{2{\cdot}1}{\mu}\right)^{5/3} \rho_c{}^{5/3} - 2{\cdot}06\ 10^{35} \frac{(1-\beta_1)^{5/3}}{\beta_1{}^{8/3}} \right\} + \frac{5{\cdot}18\ 10^{39}}{\beta_2} \left(\frac{1-\beta_1}{\beta_1}\right)^{8/3}. \qquad (54)$$

Thus for a given value of $(1-\beta_1)$, T_c depends on κ_2, the opacity in the degenerate core.

We first make, provisionally, the assumption that the flow of energy in the core is entirely radiative (*i.e.* κ_2 is the radiative opacity, Table I (iv)), and on this assumption find the formula for T_c.

Case (*a*).—Substituting

$$\kappa_2 = \frac{5{\cdot}02\ 10^{16}}{T_c{}^2},$$

$$T_c{}^4 = \frac{1}{T_c{}^2} \frac{L}{M\beta_2} \left\{ 2{\cdot}25\ 10^{39} \left(\frac{2{\cdot}1}{\mu}\right)^{5/3} \rho_c{}^{5/3} - 1{\cdot}04\ 10^{52} \frac{(1-\beta_1)^{5/3}}{\beta_1{}^{8/3}} \right\}$$
$$+ \frac{5{\cdot}18\ 10^{39}}{\beta_2} \left(\frac{1-\beta_1}{\beta_1}\right)^{8/3}. \qquad (55)$$

When the envelope thickness is small, it approximates to

$$T_c = 3{\cdot}62\ 10^6 \left(\frac{2{\cdot}1}{\mu}\right)^{5/18} \rho_c{}^{5/18} \left(\frac{L}{M}\right)^{1/6} \qquad (56)$$

$$= 2{\cdot}68\ 10^8 \left(\frac{\mu}{2{\cdot}1}\right)^{10/9} \left(\frac{M}{\odot}\right)^{5/9} \left(\frac{L}{M}\right)^{1/6}. \qquad (57)$$

Further, from (49) and (57),

$$\gamma = \frac{T_c(\text{All core})}{T_c(\text{No core})} = 0{\cdot}21\left(\frac{\odot}{M}\right)^{7/9}\left(\frac{L}{M}\right)^{1/6}. \tag{58}$$

For $M = \odot$ and $\frac{L}{M} = 10^{-2}$, T_c (All core) is about $\frac{1}{10}$th of T_c (No core).

Case (b).—We have already seen in Part I of this paper that for degeneracy the flow of energy by thermal conduction is important, and hence in the calculation of the central temperature we should take account of the conductive flow. Therefore, we should use in (54) the effective opacity κ_2' as given by (23). When the envelope thickness is very small, (54) approximates to (using the effective opacity)

$$T_c = 1{\cdot}59\ 10^8\left(\frac{2{\cdot}1}{\mu}\right)^{1/3}\left(\frac{L}{M}\right)^{1/2}\frac{I_1{}^{1/2}}{\rho_c{}^{1/6}} \tag{59}$$

$$= 1{\cdot}20\ 10^7\left(\frac{2{\cdot}1}{\mu}\right)^{7/6}\left(\frac{L}{M}\right)^{1/2}\left(\frac{\odot}{M}\right)^{1/3} I_1{}^{1/2}, \tag{60}$$

and further, from (49) and (60),

$$\frac{T_c(\text{All core})}{T_c(\text{No core})} = 0{\cdot}92\ 10^{-2}\left(\frac{\odot}{M}\right)^{5/3}\left(\frac{L}{M}\right)^{1/2} I_1{}^{1/2}. \tag{61}$$

For $M = \odot$ and $\frac{L}{M} = 10^{-2}$, T_c (All core) is about $\frac{1}{200}$th of T_c (No core).*

The great lowering in the central temperature brought about by the large thermal conductivity is best shown on comparing (57) and (60), which give T_c for an all-core configuration.

$$\frac{T_c\ (\text{calculated by taking into account conductivity})}{T_c\ (\text{calculated by neglecting conductivity})} = 4{\cdot}48\ 10^{-2}\left(\frac{L}{M}\right)^{1/3}\left(\frac{\odot}{M}\right)^{2/9} I_1{}^{1/2} \tag{62}$$

For $M = \odot$ and $L/M = 10^{-2}$, this ratio is about 1/20.

§ 3. *Case of* $M = 1/2\ \odot$.—In this article we make detailed numerical calculation for the case of $M = 1/2\ \odot$. ($= 10^{33}$ gms.). As $M < \mathfrak{M}$, all the configurations that exist are collapsed and, further, the maximum value of $1 - \beta_1$ is $1 - \beta_1{}^0 = 1{\cdot}43\ 10^{-2}$, as given by (41). Therefore, as an approximation where necessary we shall replace β_1, and so also † β_2, by unity. Thus we take $\beta_1/\beta_2 \sim 1$ in (37) and (38).

Table V gives the values for the various physical quantities for $M = 1/2\ \odot$ and for different values of $1 - \beta_1 = \frac{\kappa_1 L}{4\pi c G M}$ ranging from 0 to $1{\cdot}43\ 10^{-2}$.

Calculation of Table V.—For values of $\omega_3 = 2{\cdot}018$, $2{\cdot}119$, $4{\cdot}036$, $8{\cdot}073$, $20{\cdot}183$ ($\omega_3 \geqslant \omega_3{}^0$), the tables of solutions of Emden's equation and also the

* Taking $I_1 \sim 25$.

‡ As later calculations will show that $1 - \beta_2$ is very small, this approximation is justifiable, and its reasonableness is obvious.

solutions of the equations of fit have been constructed.* From these tables (for various values of ω_3), by drawing suitable curves we can find a/ξ_0, b/η_0, $af(a)$, . . . corresponding to the value of $\beta_1/\beta_2 = 1$. The values of $(1-\beta_1)$ corresponding to these ω_3 is given by (32) (noting that $\omega_3 \equiv C^{-1/2}$).† For large values of ω_3, (37) and (38) give a/ξ_0, b/η_0 to a good approximation. Putting numerical values (37) reduces to

$$\frac{\text{Thickness of gaseous envelope}}{\text{Radius of degenerate core}} = 20{\cdot}4\left(\frac{2{\cdot}1}{\mu}\right)^{8/3}(1-\beta_1)^{2/3}. \qquad (63)$$

(i) *Calculation of* r, ρ_c, ρ_m/ρ_c.—Having obtained the values for a/ξ_0, b/η_0, $af(a)$ . . ., these are at once calculated from the formulæ given in Table IV.

(ii) *Calculation of Central Temperature* T_c.—We first make provisionally the assumption that the flow of energy in the core is entirely radiative, and on this assumption calculate T_c from (55).

On substituting the value for T_c thus calculated in (9) and (18) we obtain the values for the degeneracy discriminant A_c and H_c/H_R. The interfacial temperature T_c is calculated from (29).

These calculations are given in the following table :—

TABLE V

$M = \frac{1}{2}\odot$ $\qquad q = \dfrac{L}{10^{31}}$

$C^{-1/2} \equiv \omega_3$	$1-\beta_1 = \dfrac{\kappa_1 L}{4\pi c G M}$	$\dfrac{\text{Core Rad.}}{\text{Total Rad.}}$ a/ξ_0	r_1 Total Radius, cm.	ρ_c Central Density grm./cm.^{3}	ρ_m/ρ_c	T' Interfacial Temperature	T_c Central Temperature	A_c Degeneracy Discriminant (at the Centre)	$\dfrac{H_c/H_R}{\text{Heat Flow}}$ $\dfrac{\text{Heat Flow}}{\text{Radiation Flow}}$ (at the Centre)
			$\times 10^9$	$\times 10^5$			$\times 10^7$		
∞	0	1	1·0	13·6	0·167	0	$8{\cdot}5\, q^{1/6}$	$102\, q^{-1/4}$	$1{\cdot}4\ 10^2\, q^{2/3}$
249	10^{-6}	0·998	1·0	13·6	0·167	$4{\cdot}6\ 10^5$	$8{\cdot}5\, q^{1/6}$	$102\, q^{-1/4}$	$1{\cdot}4\ 10^2\, q^{2/3}$
20·18	$1{\cdot}52\ 10^{-4}$	0·942	1·1	13·6	0·156	$2{\cdot}4\ 10^7$	$8{\cdot}5\, q^{1/6}$	$102\, q^{-1/4}$	$1{\cdot}4\ 10^2\, q^{2/3}$
8·07	$9{\cdot}51\ 10^{-4}$	0·83	1·2	13·5	0·15	$8{\cdot}3\ 10^7$	9·4 *	85 *	88 *
4·04	$3{\cdot}79\ 10^{-3}$	0·61	1·3	13·2	0·091	$21\ 10^7$	21	26	3·8
2·119	$1{\cdot}31\ 10^{-2}$	0·20	2·2	9·1	0·026	$47\ 10^7$	47	5·2	$7\ 10^{-2}$
2·018 $=\omega_3{}^0$	$1{\cdot}43\ 10^{-2}$	0	2·82	5·7	0·018	$50\ 10^7$	50	2·97	$2\ 10^{-2}$

* This value depends slightly, but in a complicated manner, on q. This value is for $q = 1$.

* Fairclough, *M.N.*, **91**, 55, 1930 ; **92**, 645, 1932 ; and M II.

† We may note that, strictly one ought to find a/ξ_0, b/η_0 . . . for the values of β_1/β_2 as actually found from (32) for the various values of ω_3. In our present case, however, these values for β_1/β_2 differ so little from unity, that the difference between the values for a/ξ_0, . . . so found and the values found for $\beta_1/\beta_2 = 1$ will be well within the limits of accuracy of the calculations presented in this paper. But for the case of $M >> \mathfrak{M}$ we meet with values for $(1-\beta_1)$ that are comparable to unity and in that case we have to find a/ξ_0, . . . for the actual values of β_1/β_2 (as given by (32)). The results for such configurations will be given in a subsequent paper.

Some interesting results are evident from the table, which shows how the physical quantities vary as we pass from a configuration which is all core to one in which the core has just vanished. As the total radius increases, the core decreases in size, the central density decreases, the central temperature increases and the difference between it and the interfacial temperature tends to be inappreciable. The H_c/H_R values show that when the core is large the energy flow is predominantly conductive, thus contradicting the assumption of pure radiative flow made above for calculating T_c, A_c and H_c/H_R. *Therefore, in the calculation of the central temperature we must take account of the conductive flow*, i.e. *we should use the "effective opacity" as given by* (23). The values for T_c, A_c and H_c/H_R obtained by using the effective opacity are given in Table VI. The value of the ratio of gas pressure to radiation pressure at the centre $\left(\frac{K\rho_c^{5/3}}{\mu^{5/3}}\Big/\frac{1}{3}aT_c^4\right)$ is also added. This is referred to in Part III.

TABLE VI

$M=\frac{1}{2}\odot$ $\qquad q=\frac{L}{10^{31}}$

$\frac{a/\xi_0}{\text{Total Radius}}$ Core Radius	T' Interfacial Temperature	T_c Central Temperature	A_c Degeneracy Discriminant (at the Centre)	$\frac{H_c/H_R}{\text{Radiation Flow}}$ Heat Flow (at the Centre)	$\frac{\text{Gas Pressure}}{\text{Radiation Pressure}}$ (at the Centre)	$\left(\frac{\beta_1}{1-\beta_1}\right)=\frac{\text{Gas Pressure}}{\text{Radiation Pressure}}$ (in the Envelope)
		$\times 10^7$				
1	0	$0{\cdot}72\ q^{1/2}$	$4{\cdot}1\ 10^3\ q^{-3/4}$	$2{\cdot}6\ 10^6\ q^2$	$6{\cdot}9\ 10^9\ q^{-2}$	...
0·998	$4{\cdot}6\ 10^5$	$0{\cdot}72\ q^{1/2}$	$4{\cdot}1\ 10^3\ q^{-3/4}$	$2{\cdot}6\ 10^6\ q^2$	$6{\cdot}9\ 10^9\ q^{-2}$	$1{\cdot}0\ 10^6$
0·942	$2{\cdot}4\ 10^7$	2·5 *	$6{\cdot}5\ 10^2$ *	$1{\cdot}9\ 10^4$ *	$5{\cdot}1\ 10^7$ *	$6{\cdot}6\ 10^3$
0·83	$8{\cdot}3\ 10^7$	8·3	$1{\cdot}0\ 10^2$	$1{\cdot}5\ 10^2$	$3{\cdot}9\ 10^5$	$1{\cdot}0\ 10^3$
0·61	$21\ 10^7$	21	26	3·8	$9{\cdot}9\ 10^3$	$2{\cdot}6\ 10^2$
0·20	$47\ 10^7$	47	5·2	$7{\cdot}1\ 10^{-2}$	$2{\cdot}0\ 10^2$	77
0	$50\ 10^7$	50	2·97	$2{\cdot}4\ 10^{-2}$	70	70

* This value depends slightly, but in a complicated manner, on q. The value given is for $q=1$.

The comparison between Tables V and VI shows how the large conductivity of degenerate matter modifies the results obtained for central temperature. For the case of all core, T_c is reduced by order of ten. Fig. 2 shows how T', ρ_c and T_c vary as the size of the core changes. The curves for T' and T_c, when conduction is taken into account, run together and begin only to diverge when the envelope thickness falls below—about 4 per cent. of the radius of the configuration. When conduction is neglected the two curves diverge when the thickness of the envelope falls below about 30 per cent. We see that the configuration in which the degeneracy is just setting in has the maximum central temperature (call its radius, r_0). It

should be noticed that this configuration will also have a larger T_c than any diffuse Emden-Eddington polytrope. The central temperature for an Emden-Eddington polytrope of radius r is given by *

$$T_c = \frac{1}{4}\frac{\sigma_3}{\omega_3^0}\frac{G\mu}{(R/m_H)}\frac{\beta_1^0 M}{r} = 0{\cdot}854\frac{G\mu\beta_1^0 M}{(R/m_H)r} \qquad (64)$$

$$= \frac{1{\cdot}43\ 10^{18}}{r} \quad \text{for} \quad M = 1/2\odot.$$

The radius r is arbitrary, but it must be larger than r_0 (for if it is smaller

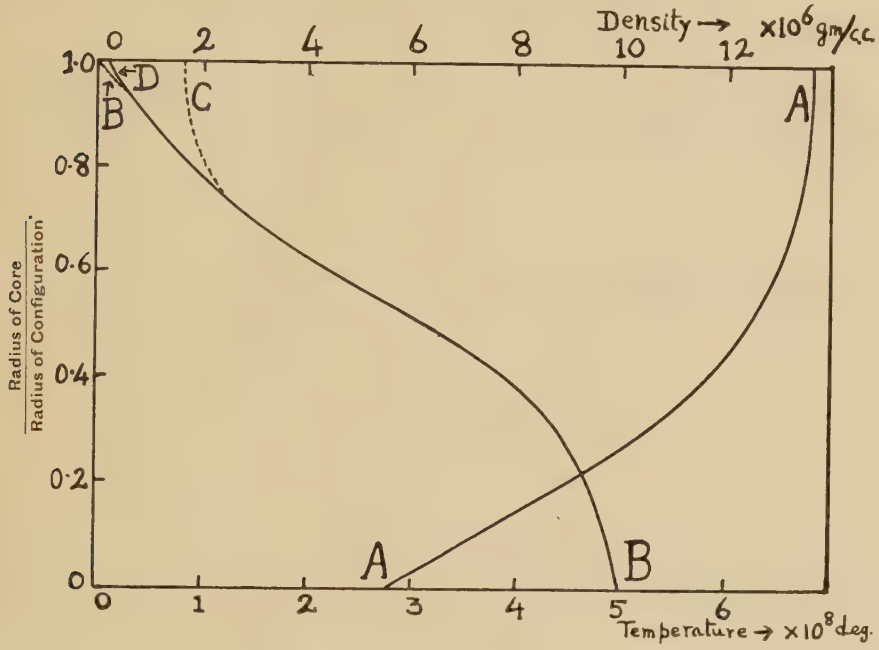

FIG. 2.—*The figure refers to Calculations made for* $M = \frac{1}{2}\odot$.

AA *represents central density* ρ_c.
BB *represents interfacial temperature* T′.
BC *represents central temperature* T_c *calculated on neglecting conduction.*
BD *represents central temperature* T_c *when conduction is considered.*

than r_0 the configuration will pass into collapsed). Thus, as the radius of the diffuse Emden-Eddington polytrope decreases, the central temperature increases, till the radius has decreased so much that degeneracy begins to set in at the centre (*i.e.* $r = r_0$). The configuration now passes into "collapsed" one and a further decrease in the radius leads to a decrease in the central temperature.†

§ 4. We have tabulated above the various physical quantities for arbitrary values of $\frac{\kappa_1 L}{4\pi c G M}$. We proceed now to estimate a "reasonable" value

* *I.C.S.*, equation (99.3). It is given here in the notation of this paper.
† This statement will hold for any $M \leqslant \mathfrak{m}$.

for κ_1, the opacity in the gaseous envelope for given L and M (*i.e.* a reasonable value for $(1-\beta_1)$ for given L, M).

Putting $\beta_1 \sim 1$ and $1-\beta_1 = \dfrac{\kappa_1 L}{4\pi cGM}$ in (29) and (30) we have

$$T' = \mathfrak{J}\left(\frac{\kappa_1 L}{4\pi cGM}\right)^{2/3}, \tag{65}$$

$$\rho' = D\mu\left(\frac{\kappa_1 L}{4\pi cGM}\right), \tag{66}$$

where

$$D = \frac{(R/m_H)^4}{\frac{1}{3}aK^3} = 1{\cdot}89 \,.\, 10^7. \tag{67}$$

Now, as an approximation, we assume κ_1, the opacity in the envelope, to be equal to the non-degenerate interfacial opacity, *i.e.* according to Kramers' law we take

$$\kappa_1 = \frac{\alpha_1 \rho'}{T'^{7/2}},$$

where α_1 is a constant whose value is known from Table I.

Using (65) and (66) we obtain

$$T' = \left(\frac{D\alpha_1}{4\pi cG}\right)^{2/7}\left(\frac{L}{M}\right)^{2/7}, \tag{68}$$

$$\rho' = \frac{(R/m_H)^{3/2}}{K^{3/2}}\left(\frac{D\alpha_1}{4\pi cG}\right)^{3/7}\left(\frac{L}{M}\right)^{3/7}\mu, \tag{69}$$

$$\kappa_1 = \frac{(R/m_H)^{3/2}}{K^{3/2}}\left(\frac{4\pi cG}{D}\right)^{4/7}\left(\frac{L}{M}\right)^{4/7}\alpha_1{}^{3/7}. \tag{70}$$

Putting for α_1 from Table I,

$$T' = 4{\cdot}61 \; 10^7\left(\frac{L}{M}\right)^{2/7}, \tag{71}$$

$$\rho' = 1{\cdot}59 \; 10^4\left(\frac{\mu}{2{\cdot}1}\right)\left(\frac{L}{M}\right)^{3/7}, \tag{72}$$

$$\kappa_1 = 10{\cdot}1\left(\frac{M}{L}\right)^{4/7}, \tag{73}$$

$$1-\beta_1 = 4{\cdot}01 \; 10^{-4}\left(\frac{L}{M}\right)^{3/7}. \tag{74}$$

Having thus estimated a reasonable value for $(1-\beta_1)$, we can now enumerate "reasonable" values for the physical quantities of the model white dwarf.

Data given:

$$\begin{cases} M = 10^{33} = \frac{1}{2}\odot., \\ L = 10^{31} \text{ ergs/sec.}, \end{cases}$$

$$L/M = 10^{-2} \text{ ergs/grm. sec.},$$

Absolute Bolometric magnitude $= 11{\cdot}3$.

Derived properties: $1-\beta_1 = 5{\cdot}56\ 10^{-5}$,
$\kappa_1 = 140$,
$T' = 1{\cdot}24\ 10^{7}$,
$\rho' = 2{\cdot}21\ 10^{3}\left(\frac{\mu}{2{\cdot}1}\right)$.

$\frac{\text{Thickness of gaseous envelope}}{\text{Radius of configuration}}$	$= 0{\cdot}03\left(\frac{2{\cdot}1}{\mu}\right)^{8/3}$.
Radius	$= 1{\cdot}05\ 10^{9}\left(\frac{2{\cdot}1}{\mu}\right)^{5/3}$ cm.
Central density	$= 1{\cdot}36\ 10^{6}\left(\frac{\mu}{2{\cdot}1}\right)^{5}$ grm./c.c.
Mean density	$= 2{\cdot}20\ 10^{5}\left(\frac{\mu}{2{\cdot}1}\right)^{5}$ grm./c.c.
Central temperature	$= 1{\cdot}37\ 10^{7}$ degrees.
Degeneracy discriminant A_c	$= 1{\cdot}58\ 10^{3}$.
H_c/H_R (at centre)	$= 2{\cdot}05\ 10^{5}$.
Effective temperature	$= 10{,}600\left(\frac{\mu}{2{\cdot}1}\right)^{5/6}$ degrees.
Opacity * (degenerate) at centre κ_2	$= 267$.
Effective opacity κ_2'	$= 1{\cdot}30\ 10^{-3}$.
Ratio of opacity of degenerate gas at centre to what would be computed for classical gas at the same temperature and density.	$= 4{\cdot}41\ 10^{-3}$.

§ 5. Having ascertained these values for the "model white dwarf," we revert to the question as to how far they represent conditions in an actual white dwarf star, *i.e.* to what extent these values will be modified when we relax the two restrictions mentioned in § 1.

(i) We have confined ourselves only to two-phase configurations (non-relativistic, "classical" and degenerate gas). The central conditions found are "mild" enough for the non-relativistic statistics to hold,† and thus the

* We have obtained above $\kappa_2 > \kappa_1$ ($\kappa_2/\kappa_1 = 1{\cdot}91$), but not much significance need be attached to this result; it arises from our method of approximation. We have taken κ_1 to be the (non-degenerate) interfacial opacity and κ_2 the opacity at the centre; hence if the core were isothermal $\kappa_2/\kappa_1 = 6{\cdot}97/A_2$ (equation (6)) and, as A_0 at the interface is 2·97 (see footnote on p. 71), so $\kappa_2/\kappa_1 = 2{\cdot}35$. As in our case the core is nearly but not perfectly isothermal, so the ratio κ_2/κ_1 comes out a little less than 2·35.

† If ρ_0 be such density that for $\rho << \rho_0$, non-relativistic statistics holds, then from equation (46) we see that for any collapsed configuration the central conditions are "mild" enough for the non-relativistic degenerate statistics to apply, provided its mass M is less than M_0, where $M_0 = 4{\cdot}32\ 10^{-4}\rho_0^{1/2}\left(\frac{2{\cdot}1}{\mu}\right)^{5/2}\odot$. Now, the non-relativistic statistic holds if $\frac{h}{mc}\left(\frac{3n}{8\pi}\right)^{1/3} << 1$, *i.e.* this gives $\rho_0 = 2{\cdot}07\ 10^{6}\left(\frac{\mu}{2{\cdot}1}\right)$ and $M_0 = 0{\cdot}62\left(\frac{2{\cdot}1}{\mu}\right)^{2}\odot$. However, if we take for ρ_0 a value $(5/4)^3$ times the above (for this value of ρ, $K_1\rho^{5/3} = K_2\rho^{4/3}$), *i.e.* $\rho_0 = 4{\cdot}04\ 10^{6}\left(\frac{\mu}{2{\cdot}1}\right)$, giving $M_0 = 0{\cdot}87\left(\frac{2{\cdot}1}{\mu}\right)^{2}\odot$. [See Stoner, *M.N.*, **92**, 662, 1932.]

configuration worked out is one possible—though not necessary—in Nature. However, this restriction to two phases has no justification beyond this, that study for two phases is a necessary preliminary for the study of a combination of larger number of phases.

(ii) We have seen that the degenerate core is almost an isothermal mass with mean density very nearly one-sixth the central density, *i.e.* the physical conditions and particularly the temperature vary very little in the core, and so the assumption of uniform source-strength in the core is reasonable. As regards the envelope, because its thickness is small, we can take in the envelope $M(r) \sim M$, $L(r) \sim L$, *i.e.* $\frac{L(r)}{M(r)} \sim \frac{L}{M}$, and thus whether the energy-generating source is present or not in the envelope, the treatment of the "generalised standard model" applies. Thus the results above given for the "model white dwarf" should give a fair "insight" as regards the conditions in an actual white dwarf.

From the formula given above we could at once calculate T_c, ρ_c etc. for "model stars" of the same mass and luminosity as Sirius B, α_2 Eridani B etc., but as the order of magnitude is sufficiently illustrated by the results for the "model star" already given, so these particular cases are not detailed here.*

It may be of interest to compare our results with those of Stoner, who in a recent paper (*M.N.*, **92**, 662, 1932) has investigated the "upper limits for densities and temperatures in stars," provided that the "star has a mass below a certain critical value which depends markedly on the mean molecular weight of the material of which it is composed and also, of course, on the distribution of density." These "upper limits" for $M = 1/2 \odot$ are readily found from the curves given in Stoner's paper.

Polytropic Index 3/2 ($M = \frac{1}{2} \odot$)

Molecular Weight, μ	$M_0/\odot$, M_0 = Stoner's Critical Mass	ρ_{max}	T_{max}
1·0	3·817	4×10^4	4×10^8
2·0	0·955	3×10^6	10^9

Thus, Stoner's T_{max} is 10^2 times larger than the central temperature we have found.

Part III.—The Relation of Gas Pressure to Radiation Pressure in Degeneracy.

We shall show below that for degenerate matter, whether non-relativistic or relativistic, the gas pressure always exceeds the radiation pressure. We first take the non-relativistic case.

* In fact the calculation given here can be applied to any star of mass $M < \mathfrak{m}$.

Case (i). *Non-relativistic Case.*

For the non-relativistic degenerate case the gas pressure * is

$$p = \frac{8\pi h^2}{15m}\left(\frac{3n}{8\pi}\right)^{5/3}, \tag{75}$$

and the radiation pressure p_R is

$$p_R = \tfrac{1}{3}aT^4. \tag{76}$$

Putting for Stefan's constant a its value $\frac{8\pi^5}{15}\frac{R}{c^3h^3}$, we obtain in terms of A_0 (the degeneracy discriminant) the value for δ (the ratio of the gas pressure to radiation pressure),

$$\delta = \frac{3^{8/3}}{\pi^{19/6}2^{5/6}}\left(\frac{mc^2}{R}\right)^{3/2}\frac{A_0^{5/3}}{T^{3/2}}, \tag{77}$$

where

$$A_0 = \frac{nh^3}{2(2\pi mRT)^{3/2}}.$$

The condition for the non-relativistic statistics to hold is that

$$\left(\frac{3nh^3}{8\pi}\right)^{1/3}\frac{1}{mc} << 1, \tag{78}$$

or substituting n in terms of A_0 and T,

$$T << \frac{mc^2}{2R}\left(\frac{3\pi^{1/2}}{4}\right)^{2/3}\frac{1}{A_0^{2/3}}. \tag{79}$$

Thus it follows from (77) and (79) that

$$\delta >> \frac{3^{5/3}2^{8/3}}{\pi^{11/3}}A_0^{8/3} >> 0{\cdot}60A_0^{8/3}. \tag{80}$$

Hence $\delta >> 1$, provided $A_0 > 1{\cdot}2$.

For degeneracy $A_0 >> 1$, hence for degeneracy $\delta >> 1$, *i.e.* the gas pressure predominates over the radiation pressure. We now treat the relativistic case.

Case (ii). *Relativistic Case.*

For the relativistic degenerate case the gas pressure is

$$p = \frac{2\pi hc}{3}\left(\frac{3n}{8\pi}\right)^{4/3} \tag{81}$$

and we obtain for δ in this case

$$\delta = \frac{5\times 3^{7/3}}{\pi^4 2^{2/3}}A_0^{4/3} = 0{\cdot}42A_0^{4/3}, \tag{82}$$

where for the relativistic case A_0 is

$$A_0 = 16\pi\left(\frac{ch}{R}\right)^3\frac{n}{T^3}. \tag{83}$$

* The formula for p, A etc. required here are given in *Phil. Mag.*, **12**, 675, 1931.

Hence $$\delta > 1, \quad \text{provided} \quad A_0 > 1{\cdot}9. \tag{84}$$

For degeneracy $A_0 >> 1$, hence $\delta >> 1$, *i.e.* the gas pressure exceeds the radiation pressure for degenerate matter (whether non-relativistic or relativistic) and for sufficiently intense degeneracy the radiation pressure is negligible when compared to the gas pressure.

Table VI, Column (6), for the case of the "model white dwarf," illustrates this result. This ratio of the gas pressure to the radiation pressure in the non-degenerate envelope of the generalised standard model depends on L, M and κ, and is given by $\delta = \dfrac{\beta_1}{1-\beta_1}$. For any two-phase configuration of mass $M \leqslant \mathfrak{M}$, β_1 does not exceed * $\frac{4}{5}$ and hence for any such configuration $\delta \geqslant 4$ (in the envelope).

For configurations of mass $M >> \mathfrak{M}$, δ may be much smaller than unity.

Part IV.—Viscosity and Degeneracy

The formulæ for the gas viscosity obtained in Paper D I are

$$\eta_N = \frac{2^{15/2} m^{1/2} R^{5/2}}{15\pi^{1/2} e^4} \frac{T^{5/2}}{ZI_2} \quad \text{for non-degeneracy}$$

$$= 3{\cdot}33 \ 10^{-17} \frac{T^{5/2}}{I_2}. \tag{85}$$

$$\eta_D = \frac{1}{15}\left(\frac{3}{8\pi}\right)^{5/3} \frac{h^5}{e^4 m^2 m_H{}^{5/3}} \frac{\rho^{5/3}}{\mu^{5/3} ZI_2} \quad \text{for degeneracy}$$

$$= 2{\cdot}65 \ 10^{-5} \frac{\rho^{5/3}}{I_2}. \tag{86}$$

$$\frac{\eta_D}{\eta_N} = \frac{1}{24}\left(\frac{3\pi^{1/2}}{4}\right)^{8/3} A_0{}^{5/3} = 8{\cdot}90 \times 10^{-2}, \tag{87}$$

where

$$I_2 = 2\pi\left[\log_e (1+\alpha_0{}^2) + \frac{1}{1+\alpha_0{}^2}\right], \tag{88}$$

$$\alpha_0 = 8{\cdot}45 \ 10^{-7} \frac{T}{\rho^{1/3}} \quad \text{for non-degeneracy}$$

$$= 5{\cdot}55 \ 10^{-2} \rho^{1/3} \quad \text{for degeneracy.}$$

The values of I_2 in the degenerate case for different values of ρ are given in Table II. It will be seen from (85), (86) and (87) that :—

(i) A *degenerate gas has greater viscosity than a classical gas at the same temperature.*

(ii) The viscosity of a degenerate gas is very large compared with what would be computed for a classical gas at the same temperature.

* M II, p. 631.

(iii) The viscosity of a classical gas is very large compared with what would be computed for a degenerate gas at the same density.

Radiative Viscosity η_R.—As first pointed out by Jeans, this arises because of the momentum carried by the radiation flowing through the star,

$$\eta_R = \frac{2aT^4}{15c\kappa\rho}. \qquad (89)^*$$

Substituting for κ the opacity coefficient from Table I, we obtain for non-degeneracy

$$\eta_R = \frac{21\sqrt{3}}{4\pi^2}\frac{\theta_1}{\theta_2}\frac{a(2\pi m)^{3/2}m_H{}^2R^{7/2}}{e^6h^2}\frac{A'\mu}{Z^2}\frac{T^{15/2}}{\rho^2} \qquad (90)$$

$$= 8{\cdot}03\ 10^{-50}\frac{T^{15/2}}{\rho^2},$$

and for the case of degeneracy,

$$\eta_R = \frac{\sqrt{3}}{28}\frac{ahR^2m_H}{e^6}\frac{A'}{Z^2}\frac{T^6}{\rho}$$

$$= 6{\cdot}77\ 10^{-43}\frac{T^6}{\rho}. \qquad (91)$$

We can now compare the radiative and gas viscosity.

For the non-degenerate case,

$$\frac{\eta\ (\text{gas})}{\eta_R} = 4{\cdot}15\ 10^{32}\frac{\rho^2}{I_2T^5}$$

$$= 1{\cdot}20\ 10^{17}\frac{A_0{}^2}{I_2T^2} \quad \text{in terms of } A_0, \qquad (92)$$

and for the degenerate case

$$\frac{\eta\ (\text{gas})}{\eta_R} = 3{\cdot}90\ 10^{37}\frac{\rho^{8/3}}{I_2T^6}$$

$$= 7{\cdot}46\ 10^6\frac{A_0{}^{8/3}}{T^2} \quad \text{in terms of } A_0. \qquad (93)$$

Thus we see that the gas viscosity predominates over the radiative viscosity when the degeneracy is intense.

The applications of these results to rotating stars and also to the origin † of the solar magnetic field will be discussed in another paper.

Part V.—Electrical Conductivity and Opacity

The present part is concerned with the connexion between absorption of light and electrical conductivity. We evaluate, by the usual application

* Milne, *Handbuch der Astrophysik*, Bd. iii/i, p. 248, 1930. Thomas has pointed out that the right-hand side of this equation requires a factor of two (*Quart. Journ. Math.*, Oxford, **1**, 239, 1930). According to Thomas, the radiative viscosity calculated here will need to be doubled.

† S. Rosseland, *Ap. J.*, **62**, 387, 1925.

of the electromagnetic theory of light absorption (in metals),* the opacity coefficient for non-degeneracy and degeneracy from the results previously obtained† for electrical conductivity. The values so calculated are in "reasonable" agreement with those obtained in the ordinary way from the method of Kramers and Gaunt. This, therefore, constitutes an interesting verification of the formulæ for electrical conductivity σ, and as the thermal conductivity λ has been calculated by exactly the same method as σ, it is also indirectly a verification for λ.

The present note is only a very rough attempt and hence many interesting points that arise are passed over. It is hoped to treat this problem in detail later.

The formulæ for σ are quoted below from D I.

For non-degeneracy,

$$\sigma_N = \frac{2^{9/2} R^{3/2} T^{3/2}}{\pi^{1/2} e^2 m^{1/2} Z I_1} = 1{\cdot}15\ 10^8 T^{3/2}/I_1 \quad \text{(E.S. Units)}; \tag{94}$$

and for degeneracy,

$$\sigma_D = \frac{3}{8\pi} \frac{h^3}{e^2 m^2 m_H} \frac{\rho}{\mu Z I_1} = 1{\cdot}67\ 10^{15} \rho/I_1 \quad \text{(E.S. Units)}, \tag{95}$$

where R is Boltzman's constant, Z the charge of the positive ion, μ the mean molecular weight, and I_1 stands for

$$I_1 = 2\pi \log_e (1 + \alpha_0^2) \tag{96}$$

$\alpha_0 = 8{\cdot}45\ 10^{-7} T/\rho^{\frac{1}{3}}$ for non-degeneracy

and

$\alpha_0 = 5{\cdot}55\ 10^{-2} \rho^{\frac{1}{3}}$ for degeneracy.

The numerical values wherever given refer to the case of fully ionised iron atoms $Z = 26$, $\mu = 2{\cdot}1$. The values of I_1 and α_0 in the degenerate case for different values of ρ are given in Part I, Table II. We have $\frac{\sigma_D}{\sigma_N} = \frac{3\pi}{32} A_0$, where A_0 is the Sommerfeld degeneracy discriminant, and further, from the values of λ given in Part I, Table I, we find

$$\frac{\lambda_N}{\sigma_N} = 4\frac{R^2}{e^2} T, \qquad \frac{\lambda_D}{\sigma_D} = \frac{\pi^2}{3} \frac{R^2}{e^2} T. \tag{97}$$

* According to the electromagnetic theory absorption occurs in all media which are not perfect dielectrics. The electric currents generated by the electric vector in the incident radiation produce heat, the energy of which must come from the energy of the incident radiation. See Drude, *The Theory of Optics* (1907), chap. iv.

† Kothari, *Phil. Mag.*, **13**, 361, 1932. This paper is referred to here as D I.

The two well-known relations obtained from the electromagnetic theory of absorption,* when we consider only free electrons (their number being n per unit volume), are

$$\mu_\nu^2(1-\alpha_\nu^2) = 1 - \frac{4\pi(m/e^2)n}{\left(\frac{n}{\sigma}\right)^2 + \left(\frac{2\pi m\nu}{e^2}\right)^2}, \tag{98}$$

$$\mu_\nu^2\alpha_\nu = \frac{1}{\nu}\frac{n^2/\sigma}{\left(\frac{n}{\sigma}\right)^2 + \left(\frac{2\pi m\nu}{e^2}\right)^2}, \tag{99}$$

where μ_ν is the index of refraction and α_ν in terms of κ_ν the mass-absorption coefficient for the frequency ν is

$$\alpha_\nu = \frac{c}{4\pi\nu}\frac{\kappa_\nu}{\mu_\nu}\rho. \tag{100}$$

When

$$\frac{2\pi m\nu}{e^2} >> \frac{n}{\sigma},$$

or substituting the values for σ, when

$$\nu >> 1{\cdot}71\ 10^{15}A_0I_1 \quad \text{for non-degeneracy,}$$
$$\nu >> 6{\cdot}89\ 10^{15}I_1 \quad \text{for degeneracy,}$$

then the relations (98) and (99) approximate to

$$\mu_\nu^2(1-\alpha_\nu^2) = 1 - \frac{ne^2}{\pi m\nu^2}, \tag{101}$$

$$\mu_\nu^2\alpha_\nu = \frac{1}{\nu^3}\left(\frac{e^2n}{2\pi m}\right)^2\frac{1}{\sigma}. \tag{102}$$

Substituting for α_ν in terms of κ_ν, we have

$$\mu_\nu\kappa_\nu = \frac{1}{\pi}\left(\frac{e^2}{m}\right)^2\frac{1}{\nu^2}\frac{n}{\mu m_H c\sigma}. \tag{103}$$

Equation (101) shows that for sufficiently large n, μ_ν will differ appreciably from unity. But as the opacity coefficient derived after the method of Kramers and Gaunt does not take account of the "refractive index," we shall also for the present take $\mu_\nu \sim 1$ in (103).

(i) *Non-degenerate case.*

Putting the value for σ in (103) we have

$$\kappa_\nu = 8{\cdot}7 \times 10^{-9}\frac{e^4}{\pi m^2 m_H{}^2 c\mu^2}\frac{I_1}{\nu^2}\frac{\rho}{T^{3/2}}. \tag{104}$$

* Drude, *The Theory of Optics* (1907), chap. v.

Taking the Rosseland mean

$$\frac{1}{\kappa}=\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial I_\nu}{\partial T}\partial\nu \Big/ \int_0^\infty \frac{\partial I_\nu}{\partial T}\partial\nu,$$

$$I_\nu=\frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/RT}-1},$$

we obtain

$$\kappa=3{\cdot}9\ 10^{22}\frac{I_1\rho}{T^{7/2}}, \tag{105}$$

where I_1 is given by (96). [For $\rho=10^3$, $T=10^7$, $I_1=3{\cdot}4$.]

(ii) *Degenerate case.*

From (95) and (103)

$$\kappa_\nu=\frac{e^4}{\pi m^2 m_H{}^2 c\mu^2}\cdot\frac{I_1}{1{\cdot}67\ 10^{15}}\frac{1}{\nu^2} \tag{106}$$

On taking the Rosseland mean,

$$\kappa=3{\cdot}7\ 10^{15}\frac{I_1}{T^2}. \tag{107}$$

From Table II (Part I), I_1 is of the order 10 for densities of the order 10^4 to 10^6, and so

$$\kappa\sim\frac{10^{16}}{T^2}. \tag{108}$$

The values of κ obtained here are as regards order of magnitude in agreement with those given in Table I (Part I), which have been derived from Kramers-Gaunt results.

Note.—In this connection it is interesting to note that Lorentz's method,* by which the conductivity in the classical case is deduced by calculating the emission in the *region of long wave-lengths* arising from the encounter of electrons with atoms, can be at once extended to include the degenerate case. Following Lorentz it is easily found that $a_0(E;\ \nu)$, the rate of absorption of energy from radiation of frequency ν and unit intensity by electrons of energy E and unit mean density is $\frac{2^{5/2}\pi e^2 l(E)E^{1/2}}{3m^{1/2}h\nu c}$, where $l(E)$ is the length of the mean-free-path for electrons with energy E.

Multiplying $a_0(E;\ \nu)$ by Rosseland's factor $(1-e^{-h\nu/RT})$ to take account of the fact that "the stimulated emission lies in the same direction as the incident radiation," the volume absorption coefficient γ is given by †

$$\gamma=\int_0^\infty a_0(E;\ \nu)(1-e^{-h\nu/RT})n(E)\,.\,dE\left[1-\frac{1}{\frac{1}{A}e^{E/RT}+1}\right], \tag{109}$$

* Lorentz, *Theory of Electrons* (reprint, 1923), pp. 82–88.

† *Astr. Nach.*, 244, 66, 1931.

where the number of electrons in the energy range E, $E+dE$ and per unit volume is

$$n(E)dE = \frac{4\pi(2m)^{3/2}E^{1/2}dE}{h^3\left(\frac{1}{A}e^{E/RT}+1\right)}. \qquad (110)$$

From (109) and (110) we obtain

$$\gamma = \frac{64\pi^2e^2RTm}{ch^3}\cdot V_0, \qquad (111)$$

where V_0 stands for the integral

$$V_0 = \int_0^\infty \frac{\frac{\partial}{\partial x}\{xl(x)\}}{\frac{1}{A}e^x+1}dx, \qquad x = \frac{E}{RT}. \qquad (112)$$

Further, from (99), for very large wave-lengths the absorption coefficient is equal to $\frac{4\pi\sigma}{c}$, and hence equating this to γ we have

$$\sigma = \frac{16\pi e^2mRT}{h^3}V_0, \qquad (113)$$

which as shown by Sommerfeld * works out to

$$\sigma_N = \frac{4}{3}\frac{e^2ln}{\sqrt{2\pi mRT}} \quad \text{(non-degeneracy)}, \qquad (114)$$

$$\sigma_D = \frac{8\pi}{3}\frac{e^2l(\bar{v})}{h}\left(\frac{3n}{8\pi}\right)^{2/3} \quad \text{(degeneracy)}, \qquad (115)$$

where

$$\bar{v} = \frac{h}{m}\left(\frac{3n}{8\pi}\right)^{1/3}. \qquad (116)$$

APPENDIX

It may be of interest to compare the physical properties of degenerate matter at the centre of the "model white dwarf" with the properties of terrestrial material, say silver. The enormous thermal conductivity of degenerate matter is particularly to be noted.

For degenerate matter at the centre of the "model white dwarf" we have found,

$$\rho = 13{\cdot}6 \times 10^6 \text{ grm./cm.}^3.$$
$$T = 1{\cdot}37 \times 10^7.$$
$$A_0 = 1{\cdot}58 \times 10^3.$$

and these values are used in the calculations given in the following table.

Maximum-energy-wavelength (according to Wien's displacement law),

$$\lambda_m = 2{\cdot}12 \text{ angstroms.}$$

* Sommerfeld, *Zs. f. Physik.*, 47, 1, 1928.

	For Degenerate Matter at Centre of "Model White Dwarf"	Ratio of Degenerate Value to Non-degenerate Value *			For Silver
Opacity (radiative)	267	$\frac{7\times 21\theta_1}{2\pi^2\theta_2}\cdot\frac{1}{A_0}$	$=$	$\frac{6\cdot 97}{A_0}$ $=4\cdot 41\times 10^{-3}$	320 for $\lambda=1\cdot 76$ A.‡ 590 for $\lambda=2\cdot 25$ A.
Electrical conductivity (ohm)$^{-1}$	$1\cdot 1\times 10^{8}$	$\frac{3\pi}{32}A_0$	$=$	$0\cdot 295A_0$ $=4\cdot 7\times 10^{2}$	$6\cdot 1\times 10^{5}$ (at 18° C.)
Thermal conductivity (C.G.S.)	$4\cdot 4\times 10^{14}$	$\frac{\pi^3}{128}A_0$	$=$	$0\cdot 242A_0$ $=3\cdot 8\times 10^{2}$	$4\cdot 1\times 10^{7}$ (at 18° C.)
Viscosity (C.G.S.)	$2\cdot 0\times 10^{4}$	$\frac{1}{24}\left(\frac{3\pi^{1/2}}{4}\right)^{8/3}A_0^{5/3}$	$=$	$8\cdot 90\times 10^{-2}A_0^{5/3}$ $=1\cdot 9\times 10^{4}$	§
Compressibility † (C.G.S.)	$1\cdot 2\times 10^{-21}$	$\left(\frac{6}{\pi}\right)^{1/3}\frac{1}{A_0^{2/3}}$	$=$	$\frac{1\cdot 24}{A_0^{2/3}}$ $=9\cdot 15\times 10^{-3}$	$0\cdot 84\times 10^{-12}$
Specific heat per electron (C.G.S.)	$4\cdot 1\times 10^{-18}$	$\left(\frac{2^{4/5}\pi}{3}\right)^{5/3}\frac{1}{A_0^{2/3}}$	$=$	$\frac{2\cdot 52}{A_0^{2/3}}$ $=2\cdot 0\times 10^{-2}$	$4\cdot 2\times 10^{-16}$ per atom at 0° C.

* Ratio of the value for degenerate gas compared with what would be computed for a classical gas at the same density and temperature.

† Isothermal compressibility $=\frac{1}{V}\frac{\partial V}{\partial p}=\frac{9}{8\pi}\frac{m}{h^2}\left(\frac{8\pi}{3n}\right)^{5/3}$.

‡ Compton, *X-rays and Electrons*, p. 184, 1927.

§ Viscosity of glacier ice $=12\times 10^{13}$; for water at 0° C. $=0\cdot 018$.

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THE ELECTRICAL CONDUCTIVITY OF AN IONISED GAS IN THE PRESENCE OF A MAGNETIC FIELD.

T. G. Cowling, M.A., D.Phil.

1. It is a well-known experimental fact that the presence of a transverse magnetic field causes certain anomalies in the flow of electric currents. A current (the Hall current) tends to flow in a direction perpendicular both to the original current and to the magnetic field, and the conductivity of the material is reduced. These phenomena are of importance in connection with the magnetic field of sunspots, and other solar and terrestrial phenomena. We shall here evaluate the magnitude of the effects in an ionised gas.

In the presence of a magnetic field an electron describes a spiral path. If the mean free path of electrons is so large that in the time between two successive collisions an electron is, in general, able to spiral several times

round the lines of force, the net distance through which it moves perpendicular to the magnetic field during a free path is small compared with the distance through which it would have moved in a similar direction were the field absent. Thus any motion across the lines of force will be greatly impeded. This explains the reduction in conductivity due to the field. On the other hand, if perpendicular electric and magnetic fields are present, in addition to their spiral motion electrons also possess a motion of drift perpendicular to both fields. The uncompensated part of this drift produces the Hall current.

Considerable difficulties appear in an application of the methods of the kinetic theory of gases to this problem. In an ionised gas, the major part of the interaction between electrons and ions is due to electrostatic forces. These forces fall off much less rapidly with distance than the forces of interaction between molecules of un-ionised gases. In consequence, the use of Boltzmann's equation, in the derivation of which it is assumed that during a large part of the motion of a gas-particle its velocity is not appreciably affected by the forces of interaction, may introduce some inaccuracy. It seems, however, legitimate to suppose that the inaccuracies resulting from a use of this equation will not affect the order of magnitude of the results, and it will accordingly be used.

2. It is sufficient to consider a gas composed of two types of particles, ions * of mass m_i and charge $+e$, and electrons of mass m_e and charge $-e$, and to suppose that the densities of the components and the temperature and mean motion of the gas are uniform throughout the gas. Let the intensities of the electric and magnetic fields be denoted by the vectors $\mathbf{E}_0$, $\mathbf{H}$. The number of electrons per unit volume is denoted by n_e, and the number per unit volume whose component velocities u_e, v_e, w_e lie in ranges du_e, dv_e, dw_e by $f_e du_e dv_e dw_e$; the quantities n_i and f_i referring to ions are similarly defined. The vectors whose components are (x, y, z), (u_j, v_j, w_j) (where j is written for either i or e), are denoted by $\mathbf{r}$ and $\mathbf{c}_j$ respectively, and $du_j dv_j dw_j$ is denoted by $d\mathbf{c}_j$.

Boltzmann's equation for f_j is of the form

$$\frac{\partial f_j}{\partial t}+\mathbf{c}_j \cdot \frac{\partial f_j}{\partial \mathbf{r}}+\frac{e_j}{m_j}(\mathbf{E}_0+\mathbf{c}_j \wedge \mathbf{H}) \cdot \frac{\partial f_j}{\partial \mathbf{c}_j}=-\sum_k \iiint (f_k f_j - f_k' f_j') g_{jk} b db d\epsilon d\mathbf{c}_k, \quad (1)$$

the quantities g_{jk}, b, ϵ, f_k', f_j' being as defined in an earlier paper,† and the expressions $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \wedge \mathbf{b}$ denoting the scalar and vector products of the vectors $\mathbf{a}$ and $\mathbf{b}$. The solution of this equation is obtained by methods similar to those used in the earlier paper. Let $\mathbf{c}_0$ denote the mean velocity of the gas; that is, let $\mathbf{c}_0$ be such that the momentum per unit volume of the gas is the same as if every particle possessed the velocity $\mathbf{c}_0$. Let us set $\mathbf{C}_j = \mathbf{c}_j - \mathbf{c}_0$, so that $\mathbf{C}_j$ is the velocity of thermal agitation of a particle; and let F_j denote Maxwell's function:

$$F_j = n_j \left(\frac{m_j}{2\pi k T}\right)^{3/2} \exp\left(-\frac{m_j C_j^2}{2kT}\right). \quad (2)$$

* Except where otherwise stated, electromagnetic units are used.

† *M.N.*, 92, 407, 1932.

Then the solution is of the form

$$f_j = F_j(1+\chi_j), \tag{3}$$

where χ_j is small. Therefore, in substituting from (3) into (1), we neglect the products $\chi_j\chi_k$, $\chi_j'\chi_k'$ on the right-hand side, and the terms involving χ_j on the left-hand side, with the exception of that in which χ_j appears multiplied by $\mathbf{H}$ and $\mathbf{C}_j$. On performing the substitution, and noting that $F_jF_k = F_j'F_k'$, we obtain the equation

$$\frac{\partial F_j}{\partial t} + \mathbf{c}_j \cdot \frac{\partial F_j}{\partial \mathbf{r}} + \frac{e_j}{m_j}(\mathbf{E}_0 + \mathbf{c}_j \wedge \mathbf{H}) \cdot \frac{\partial F_j}{\partial \mathbf{c}_j} + \frac{e_j}{m_j} F_j(\mathbf{C}_j \wedge \mathbf{H}) \cdot \frac{\partial \chi_j}{\partial \mathbf{c}_j}$$
$$= -\sum_k \iiint F_jF_k(\chi_j + \chi_k - \chi_j' - \chi_k')g_{jk}bdbd\epsilon d\mathbf{c}_k. \tag{4}$$

The function F_j depends on t only by reason of the dependence of $\mathbf{c}_0$ on t, and does not depend on $\mathbf{r}$. Now the equation of motion of the gas is

$$\rho\frac{\partial \mathbf{c}_0}{\partial t} = \sum_j n_je_j(\mathbf{E}_0 + \bar{\mathbf{c}}_j \wedge \mathbf{H}),$$

where $\bar{\mathbf{c}}_j$ denotes the mean value of $\mathbf{c}_j$, and ρ is the density of the gas. We assume that the gas is electrically neutral: thus

$$\Sigma n_je_j = 0, \tag{5}$$

and so, as $\bar{\mathbf{c}}_j = \mathbf{c}_0 + \bar{\mathbf{C}}_j$,

$$\rho\frac{\partial \mathbf{c}_0}{\partial t} = \Sigma n_je_j\bar{\mathbf{C}}_j \wedge \mathbf{H}. \tag{6}$$

Hence
$$\frac{1}{F_j}\frac{\partial F_j}{\partial t} = \frac{m_j\mathbf{C}_j}{kT} \cdot \frac{\partial \mathbf{c}_0}{\partial t} = \frac{m_j\mathbf{C}_j}{\rho kT}\sum_k n_ke_k\bar{\mathbf{C}}_k \wedge \mathbf{H}.$$

Also
$$\frac{1}{F_j}\frac{\partial F_j}{\partial \mathbf{c}_j} = -\frac{m_j\mathbf{C}_j}{kT}.$$

On use of these results, equation (4) takes the form

$$-\frac{F_j\mathbf{C}_j \cdot (\mathbf{E}_0 + \mathbf{c}_0 \wedge \mathbf{H})e_j}{kT} = -\frac{m_jF_j\mathbf{C}_j}{\rho kT} \cdot \left(\sum_k e_kn_k\bar{\mathbf{C}}_k \wedge \mathbf{H}\right) - F_je_j(\mathbf{C}_j \wedge \mathbf{H}) \cdot \frac{\partial \chi_j}{\partial \mathbf{c}_j}$$
$$-\sum_k \iiint F_jF_k(\chi_j + \chi_k - \chi_k' - \chi_k')g_{jk}bdbd\epsilon d\mathbf{c}_k. \tag{7}$$

3. The quantity $\bar{\mathbf{C}}_j$ satisfies the equation

$$n_j\bar{\mathbf{C}}_j = \int f_j\mathbf{C}_jd\mathbf{c}_j$$
$$= \int F_j\chi_j\mathbf{C}_jd\mathbf{c}_j. \tag{8}$$

Hence the right-hand sides of the two equations (7) are linear in the functions χ_j. As the left-hand sides are linear in the components of the vector $\mathbf{E}$, where

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{c}_0 \wedge \mathbf{H}, \tag{9}$$

it follows that the functions χ_j are also linear in these components. Thus, as χ_j is a scalar, we can set

$$\chi_j = \mathbf{A}_j \,.\, \mathbf{E}, \tag{10}$$

where $\mathbf{A}_j$ is a vector.

The vector $\mathbf{A}_j$ can depend only on the vectors $\mathbf{H}$, $\mathbf{C}_j$ and on scalars. Now, according to the theory of tensors, $\mathbf{H}$ is not a true vector in the same sense as $\mathbf{C}_j$ is, but its components are components of a skew-symmetrical tensor of second order; on the other hand, the vector product of $\mathbf{H}$ into a vector is a true vector, so that $\mathbf{H}_\wedge\mathbf{C}_j$, $\mathbf{H}_\wedge(\mathbf{H}_\wedge\mathbf{C}_j)$, etc., are true vectors. Now $\mathbf{H}_\wedge[\mathbf{H}_\wedge(\mathbf{H}_\wedge\mathbf{C}_j)] = -H^2\mathbf{H}_\wedge\mathbf{C}_j$; thus the only combinations of $\mathbf{H}$, $\mathbf{C}_j$ which are true vectors are the sums of scalar multiples of $\mathbf{C}_j$, $\mathbf{H}_\wedge\mathbf{C}_j$, $\mathbf{H}_\wedge(\mathbf{H}_\wedge\mathbf{C}_j)$. Thus we can set

$$A_j = \alpha_j\mathbf{C}_j + \beta_j\mathbf{H}_\wedge\mathbf{C}_j + \gamma_j\mathbf{H}_\wedge(\mathbf{H}_\wedge\mathbf{C}_j), \tag{11}$$

where α_j, β_j, γ_j are functions of the scalars C_j^2, H^2, only. Thus, by equations (8), (10), (11)

$$\begin{aligned} n_j\overline{\mathbf{C}}_j &= \int F_j\mathbf{E} \,.\, (\alpha_j\mathbf{C}_j + \beta_j\mathbf{H}_\wedge\mathbf{C}_j + \gamma_j\mathbf{H}_\wedge(\mathbf{H}_\wedge\mathbf{C}_j))\mathbf{C}_j d\mathbf{c}_j \\ &= \tfrac{1}{3}\int F_j C_j^2(\alpha_j\mathbf{E} + \beta_j\mathbf{E}_\wedge\mathbf{H} + \gamma_j(\mathbf{E}_\wedge\mathbf{H})_\wedge\mathbf{H})d\mathbf{c}_j. \end{aligned} \tag{12}$$

As $\overline{\mathbf{C}}_j$ is the velocity of diffusion of particles of one type relative to the whole mass, this equation shows that the velocity of diffusion has three components, parallel respectively to $\mathbf{E}$, $\mathbf{E}_\wedge\mathbf{H}$ and $(\mathbf{E}_\wedge\mathbf{H})_\wedge\mathbf{H}$.

To interpret this result we proceed as follows. Suppose first that $\mathbf{E}$ is parallel to $\mathbf{H}$; then $\mathbf{E}_\wedge\mathbf{H} = 0$, and so, by (10), (11),

$$\chi_j = \alpha_j\mathbf{E} \,.\, \mathbf{C}_j,$$

the terms in β_j, γ_j vanishing. Thus $\overline{\mathbf{C}}_j$ is parallel to $\mathbf{E}$, *i.e.* to $\mathbf{H}$, and

$$\overline{\mathbf{C}}_{j\wedge}\mathbf{H} = 0, \qquad (\mathbf{C}_{j\wedge}\mathbf{H}) \,.\, \frac{\partial\chi}{\partial\mathbf{c}_j} = (\mathbf{C}_{j\wedge}\mathbf{H}) \,.\, \alpha_j\mathbf{E} = 0.$$

Hence equation (7) reduces to the form

$$F_j\mathbf{C}_j \,.\, \mathbf{E}e_j/kT = \sum_k \iiint F_jF_k(\chi_j + \chi_k - \chi_j' - \chi_k')g_{jk}bdbd\epsilon d\mathbf{c}_j,$$

which is the equation valid when the magnetic field is absent. Thus the conductivity for currents in the direction of the magnetic force is unaltered by the presence of the field.

Suppose next that $\mathbf{E}$ is perpendicular to $\mathbf{H}$; then

$$\mathbf{E} \,.\, [\mathbf{H}_\wedge(\mathbf{H}_\wedge\mathbf{C}_j)] = -H^2\mathbf{E} \,.\, \mathbf{C}_j,$$

and so χ_j can be expressed in the form

$$\chi_j = \mathbf{E} \,.\, (\alpha_j\mathbf{C}_j + \beta_j\mathbf{H}_\wedge\mathbf{C}_j - \gamma_jH^2\mathbf{C}_j);$$

or, as the term in γ_j can be absorbed into the term in α_j, in the form

$$\chi_j = \mathbf{E} \,.\, (\alpha_j\mathbf{C}_j + \beta_j\mathbf{H}_\wedge\mathbf{C}_j). \tag{13}$$

Hence in this case, by (12), $\overline{\mathbf{C}}_j$ possesses two components, parallel respectively to $\mathbf{E}$ and to $\mathbf{E}_\wedge\mathbf{H}$. Thus, beside the current in the direction of the

electric force $\mathbf{E}$, there is also a current perpendicular both to the electric field and to the magnetic field; this is the Hall current.

In general, when $\mathbf{E}$ is neither parallel nor perpendicular to $\mathbf{H}$, the consequent velocity of diffusion is the vector sum of the velocities of diffusion produced by the components of $\mathbf{E}$ parallel and perpendicular to $\mathbf{H}$ when acting separately. Thus only the cases when $\mathbf{E}$ is parallel or perpendicular to $\mathbf{H}$ need be considered; and as the first case has been shown to be covered by the ordinary theory of diffusion, it is only necessary to discuss here the case in which $\mathbf{E}$ is perpendicular to $\mathbf{H}$.

4. We shall limit our work to a determination of an approximation to the velocity of diffusion similar to the first approximation of Chapman and Enskog.* This is obtained as follows. Let equation (7) be multiplied by $m_j\mathbf{C}_j$ and integrated over all values of $\mathbf{c}_j$. Then it becomes

$$-n_je_j\mathbf{E} = -\frac{n_jm_j}{\rho}\sum e_kn_k\overline{\mathbf{C}}_{k\wedge}\mathbf{H} - \int F_je_j\mathbf{C}_j(\mathbf{C}_{j\wedge}\mathbf{H}) \cdot \frac{\partial\chi_j}{\partial\mathbf{c}_j}d\mathbf{c}_j$$
$$-\sum_k\iiiint F_jF_k(\chi_j+\chi_k-\chi_j'-\chi_k')m_j\mathbf{C}_jg_{jk}bdbd\epsilon d\mathbf{c}_jd\mathbf{c}_k. \quad (14)$$

Equations (14) can be satisfied by a whole set of pairs of values of the quantities χ_j which are not solutions of (7). In particular, they possess solutions of the form (13), such that α_j, β_j are quantities independent of $\mathbf{C}_j$. These solutions can be taken as approximations to the true values of the quantities χ_j, and the value of the velocity of diffusion deduced from them is the required approximation to the true value.

We accordingly set

$$\chi_j = m_j\mathbf{D}_j \cdot \mathbf{C}_j/kT, \quad (15)$$

where $\mathbf{D}_j$ is a vector independent of $\mathbf{C}_j$. Then, by (8),

$$\overline{\mathbf{C}}_j = \mathbf{D}_j. \quad (16)$$

On substitution from (15) into (14), and on using the equation of conservation of momentum at encounter

$$m_j\mathbf{C}_j + m_k\mathbf{C}_k = m_j\mathbf{C}_j' + m_k\mathbf{C}_k',$$

the equation (14) as derived from a study of electrons takes the form

$$n_e e\mathbf{E} = -(n_em_e/\rho)\Sigma e_kn_k\mathbf{D}_{k\wedge}\mathbf{H} + en_e\mathbf{H}_\wedge\mathbf{D}_e$$
$$-\iiiint F_iF_e(\chi_i+\chi_e-\chi_i'-\chi_e')m_e\mathbf{C}_egbdbd\epsilon d\mathbf{c}_id\mathbf{c}_e. \quad (17)$$

The value of the integral in this equation is found by the methods of Chapman to be †

$$\tfrac{2}{3}\frac{(\mathbf{D}_e-\mathbf{D}_i)}{(2\pi)^{1/2}(kT)^{3/2}}I_1n_in_ee_0^4\left(\frac{m_im_e}{m_i+m_e}\right)^{1/2},$$

where I_1 is a quantity defined by Chapman (it is a slowly varying function

* Chapman, *Phil. Trans.*, **211**, 433, 1912; **216**, 279, 1916; **217**, 115, 1918. Enskog, *Upsala Dissertation*, 1917; *Archiv f. Math., Astr., och Fysik*, **16**, 1921.

† *Loc. cit.*, and *M.N.*, **82**, 292, 1922.

of ρ and T), and e_0 is the electronic charge, *measured in electrostatic units.* Also

$$-(n_e m_e/\rho)\Sigma e_j n_j \mathbf{D}_{j\wedge}\mathbf{H} + e n_e \mathbf{H}_\wedge \mathbf{D}_e = -(n_e n_i e/\rho)(m_e \mathbf{D}_{i\wedge}\mathbf{H} + m_i \mathbf{D}_{e\wedge}\mathbf{H}).$$

Thus equation (17) reduces to

$$n_e e\mathbf{E} = -(n_e n_i e/\rho)(m_e \mathbf{D}_{i\wedge}\mathbf{H} + m_i \mathbf{D}_{e\wedge}\mathbf{H}) - \tfrac{2}{3}\frac{(\mathbf{D}_e - \mathbf{D}_i)}{(2\pi)^{1/2}(kT)^{3/2}} I_1 n_i n_e e_0^4 \left(\frac{m_i m_e}{m_i + m_e}\right)^{1/2}. \quad (18)$$

The corresponding equation for ions is obtained by interchanging the suffixes i, e, and reversing the sign of the charge e. Since, by (5), $n_i = n_e$, this equation is found to be identical with (18). Thus a further equation is required to fix $\mathbf{D}_i$, $\mathbf{D}_e$. This is supplied by considerations of momentum. By the definition of $\mathbf{c}_0$,

$$(n_i m_i + n_e m_e)\mathbf{c}_0 = n_i m_i \bar{\mathbf{c}}_i + n_e m_e \bar{\mathbf{c}}_e.$$

Thus
$$n_i m_i \bar{\mathbf{C}}_i + n_e m_e \bar{\mathbf{C}}_e = 0;$$

or, on use of (16),

$$n_i m_i \mathbf{D}_i + n_e m_e \mathbf{D}_e = 0. \quad (19)$$

Now m_e/m_i is very small, and may be neglected. Thus $\mathbf{D}_i$ is negligibly small compared with $\mathbf{D}_e$. Hence in (18) we omit the terms involving $\mathbf{D}_i$, and also ignore quantities of the order m_e/m_i. Then we have

$$e\mathbf{E} = -e\mathbf{D}_{e\wedge}\mathbf{H} - \tfrac{2}{3}\frac{\mathbf{D}_e I_1 n_i m_e^{1/2} e_0^4}{(2\pi)^{1/2}(kT)^{3/2}},$$

giving as the value of $\mathbf{D}_e$,

$$\mathbf{D}_e = -(K n_i e\mathbf{E} - e^2 \mathbf{E}_\wedge \mathbf{H})/(e^2 H^2 + K^2 n_i^2), \quad (20)$$

where
$$K = 2I_1 m_e^{1/2} e_0^4/3(2\pi)^{1/2}(kT)^{3/2}. \quad (21)$$

5. The electric current-density, $\mathbf{j}$, is given by the equation

$$\mathbf{j} = \Sigma n_j e_j \bar{\mathbf{C}}_j. \quad (22)$$

Thus, on using (16) and remembering that $\mathbf{D}_i$ is negligibly small compared with $\mathbf{D}_e$, we have

$$\mathbf{j} = -n_e e \mathbf{D}_e. \quad (23)$$

Comparing (23) with (20), we see that $\mathbf{j}$ is made up of two parts: the main current, parallel to $\mathbf{E}$, and the Hall current, perpendicular both to $\mathbf{E}$ and $\mathbf{H}$. The magnitudes of these we set equal to $\sigma_1 E$ and $\sigma_2 E$ respectively. Then σ_1 denotes the ordinary conductivity of the material; σ_2 may be termed the transverse conductivity. From these definitions of σ_1, σ_2 it follows that

$$\mathbf{j} = \sigma_1 \mathbf{E} - \sigma_2 \mathbf{E}_\wedge \mathbf{H}/H, \quad (24)$$

$$\sigma_1 = \frac{n_i n_e e^2 K}{e^2 H^2 + n_i^2 K^2}, \qquad \sigma_2 = \frac{n_e e^3 H}{e^2 H^2 + n_i^2 K^2}. \quad (25)$$

By (25), the conductivity σ_1 of the material is reduced in the ratio $1 : (1 + (eH/Kn_i)^2)$ by reason of the presence of the magnetic field. Thus σ_1 is considerably affected if eH/Kn_i is of the order of unity. Since a large

reduction in σ_1 results when electrons are free to spiral about the lines of magnetic force between collisions, electrons will be free so to spiral when H is large compared with H_0, where

$$H_0 = Kn_i/e. \tag{26}$$

6. When the electric force $\mathbf{E}$ is due to a motion of the gas perpendicular to the magnetic field, $\mathbf{E} = \mathbf{c}_{0\wedge}\mathbf{H}$. On using this in (24), it follows that

$$\begin{aligned}\mathbf{j} &= \sigma_1\mathbf{c}_{0\wedge}\mathbf{H} - \sigma_2(\mathbf{c}_{0\wedge}\mathbf{H})_\wedge\mathbf{H}/H \\ &= \sigma_1\mathbf{c}_{0\wedge}\mathbf{H} + \sigma_2 H\mathbf{c}_0.\end{aligned}$$

Thus in this case the Hall current will be in the direction of $\mathbf{c}_0$; the current now arises from the fact that electrons are delayed more than ions by the magnetic field, and lag behind them. Such a process is liable to result in a separation of positive and negative charges, and in a consequent electric field.

Suppose that an electric field is set up which prevents any further flow of the Hall current. Then its intensity $\mathbf{E}_0$ is given by

$$\sigma_1\mathbf{E}_0 = -\sigma_2 H\mathbf{c}_0,$$

and the new current-density is given by

$$\begin{aligned}\mathbf{j} &= \sigma_1(\mathbf{E}_0 + \mathbf{c}_{0\wedge}\mathbf{H}) - \sigma_2(\mathbf{E}_0 + \mathbf{c}_{0\wedge}\mathbf{H})_\wedge\mathbf{H}/H \\ &= \sigma_1\mathbf{c}_{0\wedge}\mathbf{H} - \sigma_2(\mathbf{E}_{0\wedge}\mathbf{H})/H \\ &= \sigma_1(1 + \sigma_2^2/\sigma_1^2)\mathbf{c}_{0\wedge}\mathbf{H}.\end{aligned}$$

Thus the apparent conductivity in the direction of the electric force $\mathbf{c}_{0\wedge}\mathbf{H}$ is now equal to $\sigma_1(1 + \sigma_2^2/\sigma_1^2)$; that is, by (25), to $n_e e^2/n_i K$. This is the value of the conductivity when the magnetic field is absent; that is, when an electric field prevents the flow of the Hall current, the apparent conductivity in the direction of $\mathbf{c}_{0\wedge}\mathbf{H}$ is unaltered by the magnetic field.*

7. The values of the quantities σ_1, σ_2, H have been calculated for several different sets of values of n_i, T and H. The results are given in the following table. The first column of the table gives the corresponding values of P_e, the electron pressure, in dynes/cm.^{2}.

P_e	n_i	T	H_0	H	σ_1	σ_2
37·7	5×10^{13}	5500	133·4	50	$5{\cdot}228 \times 10^{-9}$	$1{\cdot}959 \times 10^{-9}$
25	$4{\cdot}562 \times 10^{13}$	4000	185·1	100	$3{\cdot}036 \times 10^{-9}$	$1{\cdot}632 \times 10^{-9}$
				2000	$3{\cdot}33 \times 10^{-11}$	$3{\cdot}6 \times 10^{-10}$
230	$1{\cdot}713 \times 10^{14}$	9800	198·1	2000	$1{\cdot}283 \times 10^{-10}$	$1{\cdot}296 \times 10^{-9}$
810	$4{\cdot}548 \times 10^{14}$	13000	341·7	2000	$6{\cdot}021 \times 10^{-10}$	$3{\cdot}524 \times 10^{-9}$
16200	$4{\cdot}513 \times 10^{15}$	26200	1171	2000	$1{\cdot}565 \times 10^{-8}$	$2{\cdot}673 \times 10^{-8}$
10^7	$1{\cdot}457 \times 10^{18}$	50000	109500	2000	$2{\cdot}116 \times 10^{-7}$	$3{\cdot}867 \times 10^{-9}$

The first set of values for n_i and T corresponds roughly to conditions in the reversing layer. The corresponding value of H is 133·4 gauss; thus the general solar field, of magnitude at most 50 gauss, is not sufficient to

* Of course, this result is not absolutely exact: an exact calculation would show that the magnetic field does produce some alteration. Compare the results for metals given by Sommerfeld, *Zeits. f. Physik*, 47, 1, 1928.

make electrons spiral freely about the lines of force at levels near to the base of the reversing layer.

8. Consider the motion of gas perpendicular to the magnetic field when no forces other than magnetic act on it. Then, by (6), (22), (24) the equation of motion is

$$\rho\frac{\partial \mathbf{c}_0}{\partial t} = \mathbf{j}\wedge\mathbf{H} = (\sigma_1\mathbf{E} - \frac{\sigma_2}{H}\mathbf{E}\wedge\mathbf{H})\wedge\mathbf{H}$$

$$= (\sigma_1\mathbf{c}_0\wedge\mathbf{H} - \frac{\sigma_2}{H}(\mathbf{c}_0\wedge\mathbf{H})\wedge\mathbf{H})\wedge\mathbf{H},$$

or, as $\mathbf{c}_0$ is perpendicular to $\mathbf{H}$,

$$\rho\frac{\partial \mathbf{c}_0}{\partial t} = -H^2\left(\sigma_1\mathbf{c}_0 - \frac{\sigma_2}{H}\mathbf{c}_0\wedge\mathbf{H}\right).$$

The solution of this vector differential equation is

$$\mathbf{c}_0 = A(\mathbf{h}\cos\omega t + \mathbf{i}\sin\omega t)e^{-t/t_0},$$

where $\mathbf{h}$ and $\mathbf{i}$ are unit vectors perpendicular to $\mathbf{H}$ and to each other, and A is an arbitrary constant, while

$$\omega = \sigma_2 H^2/\rho, \qquad t_0 = \rho/\sigma_1 H^2.$$

Thus the motion decays in a time comparable with t_0. If we set $\rho = 10^{-8}$ grs./cm.^{3}, $H = 50$ gauss and $\sigma_1 = 5\cdot10^{-9}$ E.M.U., corresponding to conditions in the reversing layer, the value of t_0 is $8\cdot10^{-4}$ secs. Thus in a time small compared with a second the motion is reduced to an inappreciable fraction of its initial value.

It follows from this that small-scale turbulent motion in the reversing layer is impossible; if any such motion is set up it is strongly damped by electromagnetic induction. On the other hand, as the currents induced in the outer layers of a large moving mass of gas tend to shield the interior from the effects of electromagnetic induction, the motion of large masses will not be so strongly damped. Let us consider the motion of a large mass of gas with a plane boundary, which moves in a magnetic field of intensity $\mathbf{H}$ with velocity $\mathbf{c}_0$; $\mathbf{H}$ is supposed to be parallel to the boundary, and $\mathbf{c}_0$ to be perpendicular to it. To shield the interior from the field, currents of strength $H/4\pi$ per unit area of the boundary must be induced in the surface layers. The consequent force on the moving mass per unit area of the boundary is of the order of $(H/2)\times(H/4\pi) = H^2/8\pi$. Thus in unit time the amount of momentum lost by the gas is that originally possessed by the gas up to a depth l below the surface, where approximately

$$H^2/8\pi = l\rho c_0,$$

or

$$l = H^2/8\pi\rho c_0.$$

It is usually assumed that the turbulent velocities existing in the solar atmosphere are of the order of 1 km./sec. Thus, if we set $c_0 = 1$ km./sec., and $H = 50$ gauss, $\rho = 10^{-8}$ grs./cm.^{3}, corresponding to conditions in the reversing

layer, we find that $l = 1$ km. Thus, if the motion of a mass of gas in the solar atmosphere perpendicular to the magnetic field is to persist for an appreciable time, the dimensions of the mass must be large compared with 1 km.*

It appears, then, that such mass-motions of gases as can exist in the solar atmosphere are large-scale effects. They are in very few respects analogous to terrestrial turbulent motion, which is a small-scale phenomenon ; they resemble rather storm-winds. This conclusion is consistent with the observational evidence indicating the presence of turbulence : for example, the granulation observed in spectroheliograms indicates a phenomenon on a very large scale.

Summary.—The conductivity of an ionised gas in the presence of a magnetic field is discussed by a method which is a generalisation of that developed by Chapman and Enskog for the discussion of problems of the kinetic theory of gases. The results are applied to a consideration of the effect of the magnetic field on gas in turbulent motion in the solar reversing layer : it is concluded that the dimensions of masses of gas in motion in the reversing layer must be large compared with 1 km.

In conclusion, I should like to thank Professor Chapman for the interest which he has shown in the progress of this work.

South Kensington :
1932 *October* 27.

THE WAVE-LENGTHS OF THE H AND K LINES OF CALCIUM IN THE ARC *IN VACUO* AND THEIR RED SHIFTS IN THE SOLAR SPECTRUM.

C. V. Jackson.

(Communicated by Professor A. Fowler, D.Sc., F.R.S.)

Introduction.—It is of considerable importance in certain astrophysical problems to have an accurate knowledge of the wave-lengths of the H and K lines of calcium when they are emitted by a source *in vacuo*, and although there have been three investigations of their wave-lengths in an arc in air, there has been hitherto none with the special object of determining their wave-lengths in the vacuum arc. Their wave-lengths in the vacuum arc have been measured interferometrically when they were present as impurity lines, but the observations were not very extensive. On account of the

* The flow of currents in a moving mass will tend to produce a space-distribution of electric charge in neighbouring regions, whose electric field opposes further flow. The consequence will be a reduction in the currents : in certain simple cases the currents are reduced by about one-half. The above conclusions are, however, unaffected by the presence of such space charges.

great importance of these lines it seemed desirable that an investigation of their wave-lengths in the vacuum arc should be specially carried out by the interferometric method.

The vacuum arc has the additional advantage, in the case of the H and K lines, of giving somewhat sharper lines than can be obtained in the arc in air.

Previous Measurements.—Before giving the results of the present investigation it will be useful to give a brief account of the former measurements. In 1910 St. John * carried out a very careful determination of the wave-lengths of H and K in the arc in air, using high dispersion grating plates, and interpolating between the interferometric iron standards of Fabry and Perot. These results were in good agreement with those of Adams,† whose values St. John was able to reduce to the same scale as his own, since Adams had given the wave-lengths of the iron lines which he used in his interpolation. In order to reduce St. John's scale to that adopted by the I.A.U. in 1928 it is necessary to subtract 0·001 A. About twenty years later Evershed ‡ redetermined the wave-lengths in the arc in air, and found values which, as can be seen in Table I, differ considerably from those of Adams and St. John, although the probable error of each investigation appears to be about the same.

In the meantime the lines had been measured interferometrically in the vacuum arc as impurities, by Burns and Meggers, § by Kiess || and by Warga.¶ Kiess made twelve observations of H and classes the measurement as A, so that this should be a very good determination ; but he observed K only four times and classes it as C. Miss Warga classes her measurements of the lines as C and D respectively, and she states that there is an unfortunate systematic difference of about 0·004 A. between her two series of observations. The final values given are the means of the two series. Her wave-lengths are therefore probably accurate to about three units in the third decimal place.

Considering the greater inherent accuracy of the interferometric method of determining wave-lengths, it is probable that the interferometer values, although they were not determinations made with the express object of finding the wave-lengths of H and K, are of about the same accuracy as the interpolated values. The results of the various observers are given in Table I. Those of St. John, Adams and Evershed have been corrected to the 1928 scale (by subtracting 0·001), and have also been reduced by 0·0017 A. to correct to vacuum arc conditions. This is the value of the pressure shift for one atmosphere, deduced from the data of Humphreys and Mohler ** and Humphreys.††

* St. John, *Astrophys. Journ.*, **31**, 156, 1910.
† Adams, quoted by St. John above.
‡ Evershed, *M.N.*, **90**, 189, 1929.
§ Burns and Meggers, quoted by Warga below.
|| Kiess, *Bureau Standards Journ. of Research R.P.*, No. 4, 1928.
¶ Warga, *Publ. Allegheny Observatory*, **6**, No. 10.
** Humphreys and Mohler, *Astrophys. Journ.*, **3**, 114, 1896.
†† Humphreys, *Astrophys. Journ.*, **26**, 18, 1907.

Table I

Observer	H	K	Notes
Adams	3968·474	3933·665	Interpolated. Corrected to 1928 scale and vacuum arc.
St. John	473	664	,, ,, ,,
Burns and Meggers	469	662	Interferometer. Vacuum arc
Kiess	469	669	,, ,, ,,
Warga	467	666	,, ,, ,,
Evershed	468	661	Interpolated. Corrected to 1928 scale and vacuum arc.
Mean	3968·470 I.A. ± 0·001	3933·664 I.A. ± 0·001	

The New Measurements.—In the present investigation the wave-lengths were determined interferometrically by comparison with the violet krypton standards, using the identical apparatus and method which were used by the author * in his work on the spectrum of the iron arc. The krypton lines † were used as a standard, rather than those of neon or cadmium, since their proximity to the H and K lines minimises the effect of any errors in the determination of the phase correction of the interferometer plates or in the scale of the ring systems. The H and K lines were produced, *in vacuo*, by burning an arc between pure carbon electrodes, on one of which was placed a minute quantity of calcium carbonate. The pressure in the arc lamp was not allowed to exceed 2 cm. *Hg*, and the current, which was taken from the 220-volt mains, was about 1·5 amperes. The calcium was present in the arc in such small quantities that only the H and K lines and 4227 appeared on the plates. Under these conditions the H and K lines are very sharp and free from self-reversal. Their width was about 0·04 A., which is about the minimum which is theoretically possible for these lines in a high temperature source, on account of the temperature Doppler broadening of spectral lines.

The photographs were taken in the equable temperature-room of the Imperial College, and the total time of exposure, for both krypton and calcium, rarely exceeded ten minutes, so that the change in length of the fused silica étalon during an exposure was not appreciable, *i.e.* less than 1/1000th of a fringe. It was found that the plates taken with the $\frac{1}{2}$-cm. étalon were the best for measurement. About thirty photographs were taken altogether, and of these fifteen were selected for measurement. Each ring system was measured at least four times, and the calculations were made to the fourth decimal place of the wave-length. The phase cor-

* Jackson, *Proc. Roy. Soc.*, A, **130**, 395, 1931.
† Krypton Standards, *Trans. I.A.U.*, 4, 1932.

rection (0·0008 A.) was applied to the mean of the fifteen observations, which gave,

H . . . 3968·470 I.A. ± 0·0006 A.
K . . . 3933·664 I.A. ± 0·0006 A.

for the wave-lengths of the lines when emitted *in vacuo* and observed in air at 15° C., 760 mm. and 50 per cent. humidity. The mean deviation in each case was just over 0·001 A., while the greatest deviations were 0·003 for both H and K. The calculated probable error was about ± 0·0003 A., but, as is well known, calculated probable errors are never very informative. The estimate given after the wave-lengths is thought to give a true idea of the actual accuracy of the wave-lengths, since it is based on the experience gained from a large number of measurements of other lines with the identical apparatus, as well as on the "probable error." It was decided not to retain the eighth figure in the wave-length, as this could not have any significant meaning in the case of lines which are 0·04 A. broad.

The result is in perfect agreement with the mean result given in Table I, which was also given to seven figures only. Additional evidence of the accuracy of the wave-lengths given above is provided by the fact that when they are used to calculate the red shift of the solar H and K lines the result is in almost perfect agreement with that found by Evershed by comparing the solar and terrestrial wave-lengths directly.

The Red Shift of the Solar H and K Lines.—The wave-lengths given for the H and K lines in the Mount Wilson Revision of Rowland's tables are 3968·494 and 3933·684 respectively, and are for the centre of the solar disc. Since they are both designated "m" they are of a high accuracy, but they must be reduced by 0·001 A. to bring them to the 1928 scale. The "Sun *minus* Vacuum Arc" displacements are then as follows :—

H . . . (R.R. corrected to 1928)—(Vac. Arc Jackson)	= 0·019 A.
K . . . „ „	= 0·023 A.
Mean (Sun – Vacuum Arc) for H and K	= 0·021 A.

This is in good agreement with the value, 0·0207 A., found by Evershed by comparing directly the solar lines with those of the arc in air after applying a correction of 0·0017 to reduce the air arc to vacuum arc.

In 1927 Evershed measured the wave-lengths of the H and K lines in the solar prominences against the I.A.U. iron standards, and by using St. John's arc wave-lengths, corrected to vacuum, he found that the mean red shift for the two lines was 0·0109 A. If the writer's new wave-lengths are used instead of those of St. John, the red shift for H and K in the prominences is 0·014 A. This is in good agreement with the mean value 0·0142 A. found by Evershed * in 1929 by direct comparison of the arc with the prominences, from a selected series of 180 spectra. (The mean for the whole series of 289 spectra was 0·0154 A., but this is of less weight than that found from the selected series.) Later, Evershed † found that the

* Evershed, *M.N.*, **90**, 189, 1929.
† Evershed, *ibid.*, **91**, 260, 1931.

shift for the limb was 0·015 A. for the mean of the H and K lines, which is in very good agreement with that found from the prominences, considering the difficulty of measuring the H and K lines at the limb of the Sun.

Summary.—The wave-lengths of the H and K lines of calcium have been measured in the vacuum arc by interferometric comparison with the new I.A.U. krypton standards, and found to be 3968·470 ± ·0006 A. and 3933·664 ± ·0006 A. These are in good agreement with the mean of the results of earlier observers, when these have been suitably corrected, and it is thought that the errors are almost certainly within the limits stated above.

The mean red shifts of the H and K lines in the solar spectrum are found to be 0·021 A. and 0·014 A. for the centre of the disc and for the prominences respectively. This is within 0·001 A. of the values found by Evershed by direct comparison of the arc and the solar lines.

In conclusion I wish to thank Professor Fowler for suggesting the research to me and for his constant interest and advice throughout the course of the work.

Imperial College, London, S.W. 7 :
1932 *October* 26.

NOTE ON THE TIDAL THEORY OF THE EVOLUTION OF THE SOLAR SYSTEM.

L. N. G. Filon, F.R.S.

In the last number of the *Monthly Notices* * Dr. Harold Jeffreys replies to a recent article by Professor F. Nölke on the origin and development of the solar system. Dr. Jeffreys refers to the difficulty of accounting, on the tidal theory, for the observed angular momenta of the Sun and planets —a difficulty which led him, in 1929, to put forward the alternative theory of an actual collision between the Sun and a star.†

This difficulty is chiefly concerned with the rotation of the planets, for Dr. Jeffreys has shown that, in the case of the Sun, the angular momentum might be reasonably accounted for by the reabsorption of ejected matter.

When, however, we come to examine this difficulty in the case of the planets, we find that it arises from a well-known theorem of hydrodynamics, which states that the circulation, $\int(udx+vdy+wdz)$, taken round any closed circuit consisting of the same particles of fluid, remains constant for all time. Dr. Jeffreys states : ‡ "So long as the conditions for the validity of this theorem are satisfied, there is no way of generating rotation

* " On the Origin of the Solar System," *M.N.*, **92**, 887, 1932.

† " Collision and the Origin of Rotation in the Solar System," *M.N.*, 89, 636–640, 1929.

‡ *Loc. cit.*, p. 637.

by the movement of solid boundaries or by bodily gravitational forces"; and, further: "We should, if the circulation theorem is valid, expect all (the planets) to have the same rate of rotation, apart from variations in radius due to condensation, and this rotation would be that of the Sun before the encounter."

It is clear that the validity of the circulation theorem is vital to the argument, and it may be useful to point out the limitations to this validity.

In the first place, it is well known that the theorem ceases to hold if the fluid is viscous. We can hardly suppose that the viscosity of the matter in the original Sun can be altogether neglected. On the other hand, it seems doubtful whether any correction obtained on these lines is likely to account for rotation of the required amount.

But there is another important exception to the validity of the theorem, one, moreover, which is almost certain to occur in a case of complicated

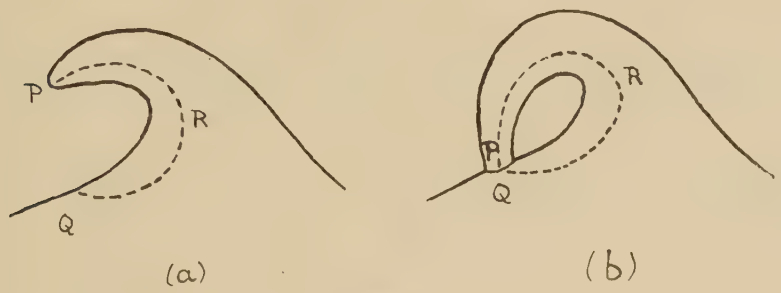

FIG. 1.

disruption originating in a tidal wave, such as is contemplated by Jeans' theory.

The circulation theorem requires the maintenance of the *continuity* of the fluid. If this is broken by separation or coalescence of parts of the fluid, the theorem fails.

This may be illustrated by the case of a "breaking" wave.

Let P be a particle in the tongue of such a wave (fig. 1, *a*), where the motion, for simplicity, is taken as two-dimensional, but the argument will apply equally to the general case. When the wave breaks, P comes into contact with a particle Q, originally widely separated from P.

Now before the fresh contact is made there will, in general, be a circulation in the *open* circuit QRP, given by $\phi_Q - \phi_P$, where ϕ is the velocity-potential at any point. The value of this circulation will obviously depend on circumstances, but there is no *a priori* reason for restricting it to an order of magnitude corresponding to velocities in the main body of the fluid (apart from the wave), velocities which may be considerably exceeded on the top of a breaking wave.

This state of things will go on until just before contact between the tip of the wave at P and the surface of the fluid about Q is actually established (fig. 1, *b*).

At this instant an impulsive pressure ϖ is brought into play over the

surface of coalescence. This impulsive pressure generates an additional velocity-distribution in the fluid (here assumed incompressible), derived from a velocity-potential $\phi' = \varpi/\rho$, ρ being the density.

The complete velocity-potential is therefore $\phi + \phi'$, ϕ being the original velocity-potential just before impact.

The space-derivative of ϕ' in the direction of the normal ν to the surface of coalescence is discontinuous, for

$$(u_\nu)_2 - (u_\nu)_1 = \left(\frac{\partial \phi'}{\partial \nu}\right)_2 - \left(\frac{\partial \phi'}{\partial \nu}\right)_1,$$

u_ν denoting the original velocity in the direction ν, and the suffixes 1, 2 referring to conditions on opposite sides of the surface of coalescence.

ϕ' itself, however, is necessarily continuous as we cross the surface of coalescence, since, by the law of action and reaction, there can be no discontinuity of ϖ.

Thus the impact, by itself, cannot introduce an additional discontinuity in the velocity-potential. The discontinuity in the total velocity-potential is therefore the original difference $\phi_Q - \phi_P$, which existed just before the impact. This, then, must be the circulation in the circuit PRQ, which is now *closed.*

If, therefore, the motion before the wave broke had been entirely irrotational and acyclic, it would no longer be acyclic after such a loop had been formed.

It is thus quite possible, if the continuity conditions are broken, for circulation to be introduced, even in a perfect fluid, without appealing to forces other than boundary pressures or body-forces derived from a potential.

In the case considered above a hollow core is formed, but if we imagine the ring, now provided with circulation, to be detached in some way, the combined effect of gravitation and viscosity must eventually cause the hollow core to be filled up, without destroying the angular momentum.

We need not, however, suppose that the angular momentum of a planet has been necessarily generated in the particular manner described.

For, if the attraction of a passing star drew a filament of matter out of the Sun, it seems possible that a large number of separate "drops" may have been formed in close proximity to one another. If one of these drops, overtaking a second one, impinged obliquely upon it, the angular momentum about their common centre of mass may well have been considerable, even if the angular momentum of each drop about its own centre of mass was quite small. Here again, owing to the violation of the continuity condition, the argument based on the circulation theorem would cease to be valid.

It would then appear that the circulation theorem does not provide a convincing argument against the tidal theory, and, until some more compelling consideration is adduced, there seems no reason why this theory should be abandoned.

THE PROPER MOTIONS OF STARS IN THE REGION OF M31.

Violet F. White, M.Sc.

Two photographic plates centred on the Andromeda nebula, M31 ($0^h\ 37^m$, $+40°\ 42'$), and with the following particulars, were obtained with the 24-inch Wilson reflector :

	Date	Exposure	Observer
1.	1894 Sept. 2	1 hour	Mr. W. E. Wilson.
2.	1929 Nov. 1	2 hours	Mr. C. C. L. Gregory.

The positions of all stars outside the nebula and within a radius 27′ of the centre of the field were measured, and proper motions were deduced in the manner described in a previous paper.* The notation used also refers to that paper.

The reliability of the values of the plate constants was determined by calculating these constants in several ways, using 36 of the fainter stars, mean magnitude 13, in all cases. Thus, values of the constants a, b, c, d, e, f, were derived from linear equations,

$$\begin{aligned} 0 &= M_x + a + bx + cy, \\ 0 &= M_y + d + ex + fy, \end{aligned} \qquad \text{(i)}$$

by (1) a least-squares solution, and (2) the method of Christie and Dyson ; these two methods gave practically identical results. A least-squares solution was also obtained using equations

$$\begin{aligned} 0 &= M_x + a + bx + cy + gx^2 + hxy + iy^2, \\ 0 &= M_y + d + ex + fy + jx^2 + kxy + ly^2, \end{aligned} \qquad \text{(ii)}$$

i.e. considering correction terms of the second order, which were assumed to be negligible in (i). The values of the constants obtained by these three methods are given in Table I.

TABLE I

	Linear Formulæ		Quadratic Formulæ			
	Least Squares	Christie and Dyson	Least Squares			
a	−0·0097	−0·0097	a	−0·0097	g	0·0001
b	0·0006	0·0006	b	0·0007	h	0·0000
c	−0·0014	−0·0014	c	−0·0014	i	−0·0001
d	−0·0154	−0·0154	d	−0·0155	j	0·0000
e	0·0025	0·0026	e	0·0025	k	−0·0001
f	0·0001	0·0001	f	0·0002	l	0·0000

* *M.N.*, 92, 519, 1932.

Hence second-order terms could be neglected, and the simplified method of Christie and Dyson gave values as accurate as the method of least squares.

The diameters of all the stellar images were measured and the magnitudes of the stars estimated. The stars were divided into two groups, comprising those with diameters equal to, or greater than, 600 scale-units, and those with diameters less than 600 scale-units (1 scale-unit $\equiv 0''{\cdot}033$). These corresponded to groups with mean magnitudes 10 and 12 respectively.

The analysis of the mean relative proper motions and mean parallaxes of the separate groups and of all the stars is given in Table II.* Two stars with large proper motions were excluded in deriving these results.

Table II

Region	Mean Mag.	n	H_m	P_x	$\bar{\mu}_x$	$P_x - \bar{\mu}_x$	P_y	$\bar{\mu}_y$	$P_y - \bar{\mu}_y$
			"						
III_{10}	10·0	57	0·013	11·3	11·7	−0·4	−7·0	−9·9	2·9
III_{12}	12·0	92	0·007	6·1	0·3	5·8	−3·8	−1·0	−2·8
III	11·2	149	0·009	7·8	4·6	3·2	−4·9	−4·4	−0·5
Comparison stars	13·0	36	0·005	4·3	0·3	4·0	−2·7	0·1	−2·8

The scale-unit in which all quantities are given, unless otherwise stated, is such that

1″ in R.A. $\equiv$ 23·0 scale-units in the x-direction (in 35 years).
„ Dec. $\equiv$ 30·3 „ „ y- „ „ „

The analysis for the comparison stars is given in the last line of Table II, and the correction (4·0, −2·8) applied to the relative proper motions to convert them to absolute proper motions was derived from this analysis.

From these absolute proper motions the numbers of stars, r, moving in the direction of the position-angle, θ, were found, and the results are tabulated in Table III.

These relations are also shown graphically by the curves in figs. 1, 2, 3. The arrows show the directions of the apices of Drifts I and II, and of the solar antapex, determined by the positions assumed in *M.N.*, **92**, 522, 1932.

As the magnitudes and parallaxes of the stars may require slight alterations, curves were also obtained directly from the relative proper motions; these are the dotted curves in the figures. Thus the variations in the curves arising from corrections (4·0, −2·8) and (0·0, 0·0) to the relative proper motions are shown.

Three complete measurements and reductions of the plates were made, and the probable errors of the measurements M_x, M_y, and of the deduced

* λ and χ in the formulæ for P_x, P_y, given in *M.N.*, **92**, 520, 1932, refer to the solar *antapex*, not the apex, as misprinted there.

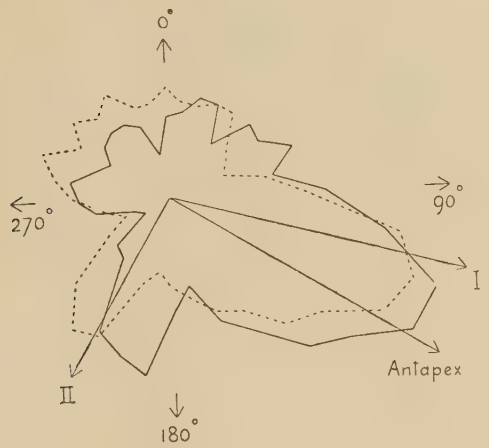

FIG. 1.—*Region III.* 149 *Stars.* *Mean Mag.* 11·2.

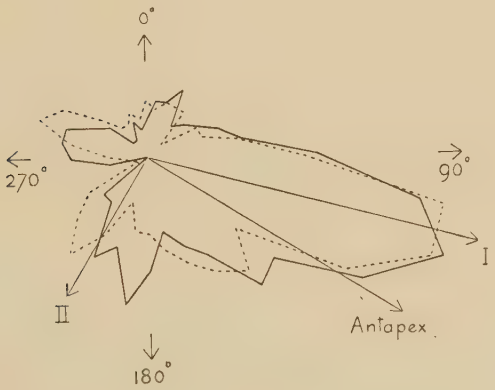

FIG. 2.—*Region III*$_{10}$. 57 *Stars.* *Mean Mag.* 10·0.

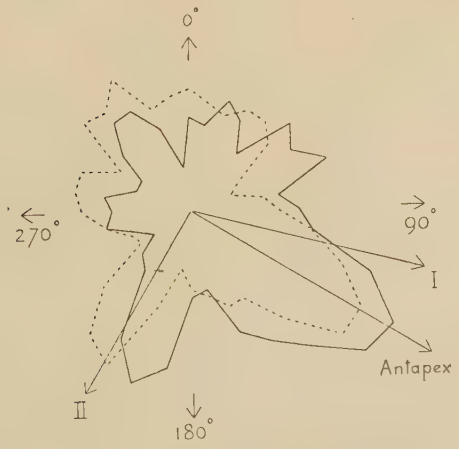

FIG. 3.—*Region III*$_{12}$. 92 *Stars.* *Mean Mag.* 12·0.

TABLE III

θ	III	III_{10}	III_{12}	θ	III	III_{10}	III_{12}
		r				r	
°				°			
0	0	0	0	180	3	3	0
10	5	1	4	190	7	2	5
20	4	2	2	200	9	3	6
30	2	0	2	210	1	0	1
40	5	2	3	220	5	3	2
50	0	0	0	230	1	0	1
60	6	0	6	240	1	0	1
70	4	2	2	250	1	0	1
80	3	1	2	260	2	0	2
90	3	1	2	270	5	2	3
100	10	6	3	280	2	2	0
110	9	6	3	290	3	0	2
120	10	3	6	300	3	2	1
130	9	3	6	310	1	0	0
140	4	2	2	320	4	0	4
150	7	4	3	330	3	0	3
160	4	0	4	340	1	0	0
170	2	1	1	350	3	1	2

proper motions μ_α, μ_δ, were calculated from the deviations of the separate quantities from their means, and gave the following results :—

	M_x	M_y	μ_α	μ_δ
	"	"	"	"
P.E. of mean per annum	± 0·0039	± 0·0042	± 0·0058	± 0·0038

Five stars in the region have had their proper motions determined from meridian observations. A comparison of the annual proper motions recorded in R. Scharr's *Eigenbewegungs-Lexikon* with the measured photographic proper motions is given in Table IV.

TABLE IV

B.D.	Mag.	Catalogue		Measured	
		μ_α	μ_δ	μ_α	μ_δ
°		s	"	s	"
39 158	7·0	−0·0028	−0·060	−0·0002	−0·034
166	8·5	−0·0005	−0·026	0·0021	−0·025
167	7·1	0·0004	−0·007	0·0037	−0·015
40 145	9·0	0·0160	−0·050	0·0152	−0·041
158	7·5	0·0078	−0·076	0·0089	−0·018

No correction has been applied to these measured proper motions to convert them to absolute proper motions ; a correction (4·0, −2·8) in μ_x, μ_y is equivalent to only ($0^s·0003$, $-0''·001$) in μ_α, μ_δ.

The number of stars in the region with proper motions, $\mu\ (= \sqrt{\mu_x^2 + \mu_y^2})$, between various limits are given in Table V.

TABLE V

μ	III	III_{10}	III_{12}
	Number of Stars		
″ ″			
0·00 – < 0·01	24	5	19
0·01 – < 0·02	47	8	39
0·02 – < 0·03	35	17	18
0·03 – < 0·04	22	16	6
0·04 – < 0·05	11	4	7
0·05 – < 0·10	9	6	3
0·10 and greater	3	3	0

I gratefully acknowledge the helpful criticisms and suggestions of Professor H. Hilton and Dr. W. M. Smart in connection with this work.

Bedford College,
London.

AN EPHEMERIS OF THE RADIANT-POINT OF THE LEONIDS.

A. King.

The present paper is based on the writer's own observations of Leonids in the years 1899–1904 and 1920–1931, and is an inquiry into the motion of the place of the radiant. Theoretically, as is well known, the radiant-point should exhibit an easterly shift of about one degree each day, but the short duration of the shower, at any rate in so far as it furnishes meteors in sufficient numbers to yield reliable radiants, coupled with the bad weather usually experienced in mid-November, has generally prevented the obtaining of the conclusive demonstration required. In the years 1928, 1930 and 1931, however, there seemed to be slight, though certain, evidences of drift, and though one might prefer to wait for more observations to confirm or disprove one's conclusions, the likelihood of "storms" of Leonids this year or next rendered it advisable to publish the results of the inquiry, in the hope that they may be thoroughly tested by accurate observation at times when the display is rich and presumably of longer effective duration. (Unfortunately in 1932 bright moonlight will militate against precise mapping.)

1. *The Observations.*—The table below contains the places of the deduced radiants. The tracks of the meteors were plotted on charts, and were afterwards transferred to a globe in determining the radiants, at the same time correcting for zenith-attraction the paths of meteors seen when the radiant was low in the sky. When the radiant was higher this correction was not applied, as being within the limits of probable error. For a similar reason the correction for diurnal aberration was disregarded; in any case, during the usual hours of observation these two corrections partly cancel each other out.

	Date G.M.A.T.	Position α	δ	No. ↓'s	Diameter of Radiant	Weight
		°	°		°	
1899	Nov. 15·62	151½	+22	7	4	1
1900	,, 15·61	151½	+22	3	3	1
1903	,, 15·86	150	+22½	10	...	1
1904	,, 15·61	150½	+21	8	...	1
1920, etc.	,, 13·97	150	+22½	7	...	1
1920	,, 15·56	152	+23	7	4⅓	2
1925	,, 15·54	150½	+23	3	...	1
1927	,, 17·56	152	+22½	4	½	2
1928	,, 15·59	150¼	+22	10	1¾	3
	,, 16·60	151¾	+22	6	2	2
1930	,, 15·58	151	+22	6	3	2
	,, 16·57	152	+21½	8	1¾	3
1931	,, 16·60	152	+22·2	22	1⅔	3
	,, 17·58	153	+22	8	4	2

The radiant for 1920 Nov. 13·97 was determined from paths of Leonids observed on November 14 in various years, and whose dates, corrected for epoch, or position in each cycle of four years beginning with leap year, averaged out at the date given. Although the diameter of some of the radiants was rather big, there was usually in these cases a well-marked concentration at the centre which enabled reliable positions to be deduced.

2. Parabolic orbits were computed for these centres as under :

	Date G.M.A.T.	L	ι	π	Ω	q	$\pi - \Omega$
		°	°	°	°		°
1899	Nov. 15·62	144·3	163·5	47·4	233·6	0·986	173·8
1900	,, 15·61	144·0	163·5	46·2	233·3	0·985	172·9
1903	,, 15·86	143·5	163·6	49·0	232·8	0·988	176·2
1904	,, 15·61	144·0	165·6	48·0	233·3	0·987	174·7
1920, etc.	,, 13·97	142·5	163·5	44·7	231·8	0·985	172·9
1920	,, 15·56	144·1	161·6	44·5	233·4	0·983	171·1
1925	,, 15·54	143·9	162·5	49·7	233·2	0·988	176·5
1927	,, 17·56	145·4	162·4	51·3	234·7	0·988	176·6
1928	,, 15·59	144·2	164·2	50·7	233·5	0·988	177·2
	,, 16·60	145·2	163·4	50·7	234·5	0·987	176·2
1930	,, 15·58	143·7	163·7	46·0	233·0	0·985	173·0
	,, 16·57	144·7	164·0	46·9	234·0	0·985	172·9
1931	,, 16·60	144·5	162·9	47·0	233·8	0·985	173·2
	,, 17·58	145·5	162·6	48·2	234·8	0·985	173·6
Weighted means		144·4	163·3	48·0	233·7	0·986	174·3

L = longitude of apex of Earth's way.

3. The dates of the showers were next adjusted for epoch, and the positions of the radiant-point corrected for precession to 1932. The weighted means were taken for each day, the radiant for 1899 November 15 being averaged with the positions for November 16, on account of the lapse of a day in 1900.

Date, 1932 G.M.A.T.	L	Mean Position α	Mean Position δ	No. of Radiants
	°	°	°	
Nov. 13·97	142·5	150·2	+22·4	1
,, 15·42	144·0	151·0	+22·2	7
,, 16·09	144·7	152·0	+21·9	4
,, 16·82	145·4	152·5	+22·2	2

There appears to be confirmatory evidence, admittedly small in extent, in the above mean positions, of the slight shift observed in 1928, 1930 and 1931 (and, it might be added, in 1920).

In 1930 the writer obtained 11 radiants of Leonids from real paths of meteors observed by members of the B.A.A. Meteor Section on November 16, and 7 similar ones on 1931 November 16. The weighted mean of these 18 radiants, corrected to 1932 November 16·5, is 152°·6+21°·8. This is very near agreement with the mean position for November 16·09 given above, especially when the shift for about 0·4 day is allowed for.

4. A consideration of the latitudes of the deduced radiants showed the value 9° 41′ as best fitting the observations. The moving radiant was therefore assumed to maintain this latitude constantly, and from this and the observed drift of about 1° per day the following ephemeris for Greenwich midnight was obtained :—

Ephemeris of the Leonid Radiant-point

Date, 1932 G.M.A.T.	Position α	Position δ	Date, 1932 G.M.A.T.	Position α	Position δ
	°	°		°	°
Nov. 12·5	148·3	+23·2	Nov. 16·5	152·3	+21·8
,, 13·5	149·3	+22·8	,, 17·5	153·3	+21·4
,, 14·5	150·3	+22·5	,, 18·5	154·3	+21·0
,, 15·5	151·3	+22·1			

The ephemeris has been extended slightly each way, inferentially, as Leonids have been seen over the period covered, if not beyond, by various observers.

Ashby, Scunthorpe, Lincs :
1932 *November* 8.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE UNIVERSITY OF LONDON OBSERVATORY.

(Communicated by the Director)

This report, concerning occultations observed at the University of London Observatory, is in continuation of one printed in *M.N.*, **91**, 949 (1931).

The error of the clock is obtained from the six-dot time signals broadcast

by the B.B.C. "Eye and ear" is the method of observation adopted by G, but a stop-watch is used by R. The two agreed within $0^s.2$ for 354 B Tauri and the mean was taken for the reduction. The three co-ordinates of the 24-inch reflector are given in *N.A.*, 1932; the refractor is within a few metres of the reflector.

TABLE I

Observations

Serial No.	Star	Mag.	Phase	Date	G.M.T.	Telescope	Observer
				1931	h m s		
6	B.D. +27° 1337	6·4	D	Mar. 27	22 22 9·12	R	R
7	Spica	1·2	R	July 21	20 8 52·49	R	R
8	136 Tau	4·6	R	Aug. 10	3 2 34·82	R	R
9	*c* Gem	5·5	R	Nov. 1	23 7 10·74	C	G
				1932			
10	354 B Tau	6·4	D	Feb. 16	19 38 9·21	R	R
11	354 B Tau	6·4	D	16	19 38 8·99	C	G
12	134 B Gem	6·5	D	18	18 21 40·31	R	R
13	ι Vir	5·7	D	May 17	20 5 28·41	R	R
14	ι Vir	5·7	D	17	20 5 27·80	C	G
15	W.Z.C. 795	7·0	D	Jun. 12	21 42 14·59	R	R
16	W.Z.C. 796	6·6	D	12	21 43 20·45	R	R
17	49 Vir	5·2	D	13	20 46 27·95	R	R
18	96 Aqu	5·7	D	Oct. 11	18 58 31·79	R	R

Observers:
C. C. L. Gregory (G).
Frank Robbins (R).

Telescopes:
8-inch refractor (R).
24-inch reflector (Cassegrain) (C).

Column 6 in Table I contains the Greenwich mean time of the phenomenon observed. In the reductions (Table II) a correction of $0^h.0030$ has been added to the G.M.T. to allow for an error of 6″ in the mean longitude of the Moon as suggested by Professor E. W. Brown. No other correction for lag, personal error, etc., has been included.

TABLE II

Reductions

No.	χ	ρ	$\sigma' - \sigma$	$\chi - \rho$	$\cos(\chi - \rho)$
	° ′	° ′		° ′	
6	82 13	96 57	− 0·2	− 14 44	+0·967
7	200 19	116 44	+ 2·2	+ 83 35	+0·112
8	208 44	87 50	(+48·5)	+120 54	−0·514
9	278 56	101 6	+ 0·1	+177 50	−0·999
10	90 24	85 1	+ 0·3	+ 5 23	+0·996
11					
12	127 12	98 36	+ 1·1	+ 28 36	+0·878

No.	χ	ρ	$\sigma' - \sigma$	$\chi - \rho$	$\cos(\chi - \rho)$
	° ′	° ′		° ′	
13	148 22	116 22	+ 0·1	+ 32 0	+0·848
14	148 21	116 22	+ 0·5	+ 31 59	+0·848
15	61 2	118 18	+ 1·7	− 57 16	+0·541
16	58 32	118 18	+ 1·7	− 59 46	+0·504
17	136 34	116 59	− 0·9	+ 19 35	+0·942
18	39 52	61 51	+ 0·5	− 21 59	+0·927

The reductions were done by R and checked by G ; the method followed is that adopted by the *Nautical Almanac* office.

University of London Observatory,
Mill Hill Park, N.W. 7 :
1932 *October* 19.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE NATIONAL OBSERVATORY OF ATHENS.

S. Plakidis and G. Adamopoulos.

Ref. No.	Star	Mag.	Date	Local Sidereal Time	Observer
	°		1932	h m s	
1	B.D. +28 1069	9·3	May 9	10 47 28·4	AD
2	+28 1070	9·1	9	10 54 18·3	,,
3	+28 1071	7·3	9	11 2 27·2	,,
4	+28 1073	8·0	9	11 26 43·0	,,
5	+27 1059	9·3	9	11 34 43·5	PL
6	+28 1074	9·3	9	11 34 59·8	,,
7	+27 1058	9·5	9	11 38 7·4	,,
8	+27 1057	9·5	9	11 39 48·3	,,
9	+27 1061	9·2	9	11 42 8·5	,,
10	+28 1076	9·5	9	11 43 29·3	,,
11	+28 1080	8·6	9	12 19 29·4	,,
12	+27 1073	9·3	9	12 21 57·3	,,
13	+28 1082	9·3	9	12 24 20·3	,,
14	+27 1068	8·7	9	12 25 22·8	,,
15	+27 1070	8·5	9	12 27 17·6	,,
16	+26 1487	7·8	10	10 40 4·1	AD
17	+26 1510	7·5	10	12 39 4·3	PL
18	+26 1508	6·5	10	12 41 36·6	,,
19	+26 1512	7·6	10	12 55 34·3	,,
20	−11 3507	8·7	17	13 10 28·7	,,
21	+26 1427	8·4	June 6	13 26 4·1	,,
22	+26 1431	8·9	6	13 31 57·9	,,
23	+26 1440	8·9	6	14 16 50·9	,,
24	+25 1794	7·5	7	13 0 44·8	AD
25	+24 1810	9·5	7	13 39 15·8	PL
26	+24 1811	8·7	7	13 41 23·8	,,
27	+24 1812	9·3	7	13 58 46·8	,,

Ref. No.	Star	Mag.	Date	Local Sidereal Time	Observer
	°		1932	h m s	
28	B.D. +24 1815	9·0	June 7	14 10 14·0	PL
29	+24 1816	9·0	7	14 13 55·0	,,
30	+24 1820	8·8	7	14 31 17·3	,,
31	+20 2233	8·5	8	14 50 47·6	,,
32	+20 2232	7·2	8	15 8 14·7	,,
33	+10 2167	9·0	10	13 35 21·2	AD
34	+10 2176	8·5	10	15 51 34·3	PL
35	+ 3 2502	7·5	11	16 15 41·3	AD
36	+ 3 2503	8·0	11	16 15 45·9	,,
37	+ 3 2504	5·0	11	16 54 7·4	,,
38	−22 4000	7·7	15	14 38 52·2	,,
39	+12 2193	8·1	July 7	15 6 58·0	,,
40	+11 2210	8·8	7	15 32 44·0	,,
41	+11 2213	9·5	7	15 46 3·1	PL
42	+11 2212	8·0	7	16 0 47·0	,,
43	+ 5 2457	9·0	8	15 54 43·9	,,
44	− 1 2600	7·7	9	15 35 48·1	AD
45 A.G. Cordoba, A	11083	8·5	13	18 15 38·2	,,
46 ,, ,, B	10466	8·0	14	17 38 22·2	,,
47 ,, ,, ,,	10489	6·5	14	18 41 17·5	,,
48 ,, ,, ,,	11328	8·1	15	17 50 54·7	,,
49 ,, ,, ,,	11329	7·9	15	17 51 35·7	,,

Instrument used: Doridis refractor (Gautier, 400 mm.).—*M.N.*, **92**, 724, 1932.

Magnifying power: 155 times.

Phase observed: Only disappearances at the dark limb.

Observers: S. Plakidis, D.Sc. (PL), G. Adamopoulos (AD).

National Observatory of Athens :
1932 *August* 30.